where $S(X,Y) = \frac{1}{2\mu}\{(X\mu)Y + (Y\mu)X - g(X,Y)\operatorname{grad}\mu\}$ and $\operatorname{grad}\mu$ is calculated in the metric g, that is,

$$X(\mu) = g(X, \operatorname{grad}\mu).$$

Hint: Since $\overline{\overline{\nabla}}$ is obviously symmetric, it suffices to show that $\overline{\overline{\nabla}}$ is compatible with $\bar{g}$, that is, that

$$X(\bar{g}(Y,Z)) = \bar{g}(\overline{\overline{\nabla}}_X Y, Z) + \bar{g}(Y, \overline{\overline{\nabla}}_X Z).$$

But the first member of the equality above is

$$X(\mu g(Y,Z)) = X(\mu)g(Y,Z) + \mu g(\nabla_X Y, Z) + \mu g(Y, \nabla_X Z),$$

and the second is

$$\mu g(\nabla_X Y, Z) + \mu g(Y, \nabla_X Z)$$
$$+\mu\{g(S(X,Y),Z) + g(Y, S(X,Z))\}.$$

Therefore, it is enough to prove that

$$X(\mu)g(Y,Z) = \mu\{g(S(X,Y),Z) + g(Y,S(X,Z))\},$$

which follows from a direct calculation.

6. (*Umbilic hypersurfaces of the hyperbolic space*). Let (M^{n+1}, g) be a manifold with a Riemannian metric g and let ∇ be its Riemannian connection. We say an immersion $x: N^n \to M^{n+1}$ is (totally) *umbilic* if for all $p \in N$, the second fundamental form B of x at p satisfies

$$\langle B(X,Y), \eta\rangle (p) = \lambda(p)\langle X, Y\rangle, \qquad \lambda(p) \in \mathbf{R},$$

for all $X, Y \in \mathcal{X}(N)$ and for a given unit field η normal to $x(N)$; here we are using $\langle\ ,\ \rangle$ to denote the metric g on M and the metric induced by x on N.

a) Show that if M^{n+1} has constant sectional curvature, λ does not depend on p.

Hint: Let $T, X, Y \in \mathcal{X}(N)$. The given condition implies that

$$-\langle\nabla_X \eta, Y\rangle = \lambda\langle X, Y\rangle \qquad \text{and} \qquad -\langle\nabla_T \eta, Y\rangle = \lambda\langle T, Y\rangle.$$

Differentiate the first equation with respect to T and the second with respect to X, obtaining, for all Y,

$$\langle \nabla_T \nabla_X \eta - \nabla_X \nabla_T \eta, Y \rangle$$

$$= - \left\langle T(\lambda)X - X(\lambda)T + \nabla_{[X,T]}\eta, Y \right\rangle .$$

Use the fact that M has constant sectional curvature to conclude that $T(\lambda)X - X(\lambda)T = 0$. Because T and X can be chosen linearly independently, this implies that $X(\lambda) = 0$, for all $X \in \mathcal{X}(N)$; therefore $\lambda = \text{const}$.

b) Use Exercise 5 to show that if we change the metric g to a metric $\bar{g} = \mu g$, conformal to g, the immersion $x: N^n \to (M^{n+1}, \bar{g})$ continues being umbilic, that is, if (using the notation of Exercise 5) $\langle \nabla_X \eta, Y \rangle_g = -\lambda \langle X, Y \rangle_g$, then

$$\left\langle \overline{\nabla}_X \left(\frac{\eta}{\sqrt{\mu}}\right), Y \right\rangle_{\bar{g}} = \frac{-2\lambda\mu + \eta(\mu)}{2\mu\sqrt{\mu}} \langle X, Y \rangle_{\bar{g}} .$$

c) Take $M^{n+1} = \mathbf{R}^{n+1}$ with the euclidean metric. Show that if $x: N^n \to \mathbf{R}^{n+1}$ is umbilic, then $x(N)$ is contained in an n-plane or an n-sphere in $\mathbf{R}^{n+1}$.

Hint: For (a), $\lambda = \text{constant}$. If $\lambda = 0$, $\langle \nabla_X \eta, Y \rangle = 0$ for all $X, Y \in \mathcal{X}(N)$ and all $\eta \in \mathcal{X}(N)^{\perp}$. It follows that $x(N)$ is contained in an affine n-plane in $\mathbf{R}^{n+1}$. If $\lambda \neq 0$, consider the map $y: N \to \mathbf{R}^{n+1}$ given by

$$y(p) = x(p) - \frac{\eta(p)}{\lambda}, \qquad p \in N.$$

Let $T, Y \in \mathcal{X}(N)$. Observe that

$$\langle \nabla_T y, Y \rangle = \langle T, Y \rangle - \frac{1}{\lambda} \langle \nabla_T \eta, Y \rangle = 0.$$

It follows that $y(N)$ reduces to a point, call it x_o, and that x satisfies

$$|x(p) - x_o|^2 = 1/\lambda^2,$$

that is, $x(N)$ is contained in a sphere of center x_o and radius $1/\lambda$.

d) Use (b) and (c) to establish that the umbilic hypersurfaces of the hyperbolic space, in the upper half-space model H^{n+1}, are the intersections with H^{n+1} of n-planes or n-spheres of $\mathbf{R}^{n+1}$. Therefore, the umbilic hypersurfaces of the hyperbolic space are the geodesic spheres, the horospheres and the hyperspheres. Conclude that such hypersurfaces have constant sectional curvature.

e) Calculate the mean curvature and the sectional curvature of the umbilic hypersurfaces of the hyperbolic space.

Hint: Consider the model of H^n as the upper half-space. Let $\sum = S \cap H^n$ be the intersection of H^n with a Euclidean $(n-1)$-sphere $S \subset \mathbf{R}^n$ of radius 1 and center in H^n. Since $\sum$ is umbilic, all of the directions are principal, and it is enough to calculate the curvature of the curves of intersection of $\sum$ with the x_1x_n-plane. Use the expression obtained in part (b) of this exercise to establish that the mean curvature of $\sum$ (in the metric of H^n) is equal to 1 if S is tangent to ∂H^n, is equal to $\cos\alpha$ if S makes an angle α with ∂H^n, and is equal to the "height" of the Euclidean center of S relative to ∂H^n, if $S \subset H^n$. To calculate the sectional curvature, use the Gauss formula.

7. Define a "stereographic projection" $f: H^n_{-1} \to D^n$ from the model of the hyperbolic space H^n_{-1} of curvature -1 given in Exercise 3 onto the open ball

$$D^n = \{(x_o, \ldots, x_n); x_o = 0, \sum_{\alpha=1}^{n} x_\alpha^2 < 1\}$$

in the following way: If $p \in H^n_{-1} \subset L^{n+1}$, join p to $p_o = (-1, 0, \ldots, 0)$ by a line r; $f(p)$ is the intersection of r with D^n (See Fig. 3). Let $p = (x_o, \ldots, x_n)$ and $f(p) = (0, u_1, \ldots, u_n)$.

a) Prove that:

$$x_\alpha = \frac{2u_\alpha}{1 - \sum_\alpha u_\alpha^2}, \quad \alpha = 1, \ldots, n,$$

$$x_o = \frac{2}{1 - \sum_\alpha u_\alpha^2} - 1.$$