Average of digits

Let $x \in (0, 1)$ be written in base $b \geq 2$. For any x such that its decimal expansion does not end in all 0s or all $(b-1)$s, write

$$x = 0.x_1x_2x_3...$$

Then define the function

$$f(x) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} x_i$$

(if such a limit exists). We will prove that in any base b, there exists a number such that $f(x) = x$.

Proof: Let a base $b \geq 2$ be given. First, we will define the auxiliary function $g(x)$. This function is similar to $f(x)$ except that its domain is $x \in (0, 1)$ such that x has a finite decimal expansion (in base b). For future use, denote the set of such rationals as S. Let this decimal expansion (in base b) be written

$$x = 0.x_1x_2x_3...x_k00000\cdots$$

That is, we take the expansion with all 0s at the end (in the future, we will not include this). Then $g(x)$ is defined to be

$$g(x) = \frac{1}{k} \sum_{i=1}^{k} x_i$$

Fact 1: For any x in S, there exists

$$y_n = 0.z_1z_2...z_n$$

such that $\lim_{n\to\infty} g(y_n) = x$. Let

$$x = 0.x_1x_2x_3...x_k$$

and define y_1 to be 0.1. Now, define z_n recursively to be

$$z_n = \begin{cases} 0 & g(0.z_1z_2...z_{n-1}) > x \\ b-1 & g(0.z_1z_2...z_{n-1}) \leq x \end{cases}$$

From this definition, it is obvious that

$$\lim_{n\to\infty} g(0.z_1z_2...z_n) = x$$

Fact 2: For any $x, y \in S$, there exists

$$z_n = 0.u_1u_2...u_n$$

such that $\lim_{n\to\infty} g(y + z_n) = x$. First, let the decimal expansion of y be

$$y = 0.y_1y_2...y_t$$

From **Fact 1**, we know there exists $w_n \in S$ such that

$$\lim_{n\to\infty} g(w_n) = x$$

Let the decimal expansion of this w_n be

$$w_n = 0.u_0u_1...u_k$$

Now, choose N large enough such that $n \geq N$ implies

$$|x - g(w_n)| < \frac{\epsilon}{2} \text{ and } \frac{b^t}{N} < \frac{\epsilon}{2}$$

With this, let

$$z_n = w_n b^{-t}$$

Then

$$y + z_n = 0.y_1y_2...y_tu_1u_2...u_n$$

and therefore (for $n \geq N$)

$$g(y+z_n) = \frac{1}{n}\left(\sum_{i=1}^{t} y_i + \sum_{i=1}^{n} u_i\right) < \frac{b^t}{n} + \frac{1}{n}\sum_{i=1}^{n} u_i < \frac{\epsilon}{2} + g(w_n)$$

$$g(y+z_n) = \frac{1}{n}\left(\sum_{i=1}^{t} y_i + \sum_{i=1}^{n} u_i\right) > \frac{0^t}{n} + \frac{1}{n}\sum_{i=1}^{n} u_i = g(w_n)$$

This leads to the final conclusion

$$|x - g(y+z_n)| < \frac{\epsilon}{2} + |x - g(w_n)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

and we are done.

Fact 3: Let y_n be a sequence in S such that

$$y_1 = 0.x_1$$

$$y_2 = 0.x_1x_2$$

$$\vdots$$

$$y_n = 0.x_1x_2...x_n$$

and let $\lim_{n\to\infty} y_n = L \notin S$. Then

$$\lim_{n\to\infty} g(y_n) = f(L)$$

(when this limit exists). Write

$$L = 0.L_1L_2L_3...$$

Since y_n approaches L, it is obvious that $x_i = L_i$ for all $i \in \mathbb{N}$. This naturally leads to the conclusion as $f(L)$ is defined to be

$$f(L) = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} L_i = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} x_i = \lim_{n\to\infty} g(y_n)$$

and we are done.

Main Conclusion: We now come to the main portion of our proof. Let $x_0 = 0.101$. From **Fact 2**, we know there exists $z_0 \in S$ such that

$$|x_0 + b^{-6} - g(x_0 + z_0)| < b^{-12}$$

Thus, define $x_1 = x_0 + z_0$ and note that $g(x_1) = x_0$ for at least the first 3 digits. This suggests a strategy:

- Let $\phi(n) = t$ where $x_n = 0.y_1 y_2 ... y_t$
- Define z_n such that $|x_n + b^{-2\phi(n)} - g(x_n + z_n)| < b^{-4\phi(n)}$
- Define $x_{n+1} = x_n + z_n$

In this manner, we have constructed a sequence x_n such that $g(x_n) = x_n$ for at least the first $\phi(n)$ digits. It is obvious that $\lim_{n\to\infty} x_n = L$. Since $g(x_n)$ agrees with x_n for an increasing number of digits, we can use **Fact 3** to conclude that

$$f(L) = \lim_{n\to\infty} g(x_n) = \lim_{n\to\infty} x_n = L$$

We conclude that there exits L such that $f(L) = L$.