

# On the algebraic topology of finite spaces

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## Abstract

We explore the properties of finite topological spaces using the methods of elementary algebraic topology. Among other results, we prove a key theorem of McCord relating finite spaces to simplicial complexes without using any machinery of homotopy theory. Few of our results are original, but most were discovered independently, and the presentation is intended to show the motivation behind these results.

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## Introduction

In July of 2007, a friend of mine posed the following little problem to me: find a finite topological space whose fundamental group is nontrivial. After solving this problem (see Section 2.3), I went on to explore other examples of finite spaces. Eventually, I realized that there is a beautiful and simple theory of the algebraic topology of finite spaces, at the heart of which lies the lifting Theorem 3.5 and its corollaries. This paper presents this theory.

The treatment of finite spaces given in this paper is not meant to be the “deepest” possible treatment or a comprehensive survey of what is known about finite spaces. Rather, it is intended to guide the reader through the results in a way that she might stumble upon them for herself. Indeed, the topics in this paper are organized loosely based on the order in which I discovered them myself. Really, for the ambitious reader, I might suggest not reading the paper at all and instead thinking about the problem above and seeing what she can prove on her own.

I make no claims as to the originality of any of the results in this paper, but I (re)discovered most of them on my own. Whenever I did not discover a significant result independently, I mention the original source with the result. I in particular am not aware of whether Theorem 3.5 has never explicitly been used to prove the basic properties of finite spaces (the paper [7] that first proved Corollary 3.7 of the Theorem took a different approach, which is deeper and more generalizable but less elementary).

I assume the reader is familiar with no more than the basic concepts of algebraic topology, all of which can be found in [6], though other books, such as [10], give better treatments of simplicial complexes. All maps between topological spaces are assumed continuous unless stated otherwise. We say two spaces  $X$  and  $Y$  are “weak homotopy equivalent” not only if there is a weak homotopy equivalence between them, but if there is a sequence of weak homotopy equivalences  $X = Z_0 \leftarrow Z_1 \rightarrow Z_2 \leftarrow \dots \leftarrow Z_{n-1} \rightarrow Z_n = Y$  connecting  $X$  and  $Y$  (by using CW models, in fact,  $n = 2$  always suffices).

## 1 Preliminaries

Finite topological spaces are pathological beasts, far removed from the spaces one normally deals with in mathematics. They do bear some resemblance to

the Zariski topologies of algebraic geometry, but in general they are unlike any spaces that we normally deal with. In this section, we get a handle on finite spaces from the perspective of pointset topology.

## 1.1 The $T_0$ quotient

First, we show that if we are interested only in the homotopy-theoretic properties of finite spaces (or indeed any spaces), we may ignore the worst possible pathology that can occur. Recall that a topological space  $X$  is  $T_0$  if for any distinct points  $x, y \in X$ , there is some open set such that either  $x \in U$  but  $y \notin U$  or  $y \in U$  but  $x \notin U$ . Given a space  $X$ , we can identify pairs of points  $x$  and  $y$  such that this fails (i.e. such that  $x \in U$  iff  $y \in U$  for any open set  $U$ ; this is clearly an equivalence relation) to obtain a quotient space  $X_0$  that is  $T_0$ .

**Proposition 1.1.** *Let  $X$  and  $X_0$  be as above and let  $\pi : X \rightarrow X_0$  be the quotient map. Then  $\pi$  is a homotopy equivalence.*

*Proof.* Let  $f : X_0 \rightarrow X$  be any map such that  $\pi f = 1$  (i.e.,  $f$  picks a representative of each equivalence class); we claim  $f$  is a homotopy inverse to  $\pi$ . It suffices to show that  $g = f\pi$  is homotopic to the identity on  $X$ . Define  $H : X \times I \rightarrow X$  by  $H(x, t) = g(x)$  for  $t = 0$  and  $H(x, t) = x$  for  $t \in (0, 1]$ . By definition of  $\pi$ ,  $g(x)$  is contained in exactly the same open sets as  $x$  is. Thus for any  $U \subseteq X$  open,  $H^{-1}(U) = U \times I$ , so  $H$  is continuous. Hence  $H$  is a homotopy from  $g$  to 1.  $\square$

We remark that the construction of  $X_0$  from  $X$  is functorial, and is in fact left adjoint to the inclusion of the full subcategory of  $T_0$  spaces in the category of all topological spaces. These facts are straightforward to verify, and their proofs are omitted here because they are tangential to our main interest. On a more whimsical note, we observe that Proposition 1.1 is actually equivalent to the axiom of choice; the proof is left as an exercise to the reader.

In light of this result, in the rest of this paper the term “space” will be taken to mean “ $T_0$  topological space.”

## 1.2 Finite spaces, locally finite spaces, and posets

We now show that finite spaces are actually essentially the same thing as more familiar objects, namely finite posets.

**Definition 1.2.** *Let  $P$  be a poset and  $x \in P$ . Then the upset and downset of  $x$  in  $P$  are respectively  $x^\uparrow = \{y \in P : y \geq x\}$  and  $x^\downarrow = \{y \in P : y \leq x\}$ . The order topology on  $P$  is the topology that has  $\{x^\downarrow : x \in P\}$  as a basis of open sets. An upset or downset in  $P$  is respectively a closed or open set in the order topology of  $P$ .*

We will always consider posets to be equipped with the order topology. We note that the antisymmetry axiom of posets is equivalent to the  $T_0$  axiom holding for the order topology. For  $x \in P$ ,  $x^\uparrow$  and  $x^\downarrow$  are respectively the minimal closed

and minimal open sets containing  $x$ , and that  $A \subseteq P$  is an upset (downset) iff  $x \in A$  implies  $y \in A$  for all  $y \geq x$  ( $y \leq x$ ).

**Theorem 1.3.** *The association of a finite poset with its order topology is an isomorphism from the category of finite posets to the category of finite spaces.*

*Proof.* Let  $X$  be a finite space. Define a partial order on  $X$  by saying  $x \preceq y$  iff  $y \in \overline{\{x\}}$ . This is clearly transitive, and is antisymmetric since  $X$  is  $T_0$ . We claim that the order topology induced by  $\preceq$  is the same as the given topology on  $X$ . By definition of  $\preceq$ ,  $x^\uparrow = \overline{\{x\}}$  so the upsets of single points are closed in the original topology. Since  $X$  is finite, every upset is the union of the upsets of finitely many single points (namely, its elements), and hence every upset is closed in the original topology. Conversely, if  $C \subseteq X$  is closed in the original topology,  $x \in C$  implies  $x^\uparrow = \overline{\{x\}} \subseteq C$ . Thus  $C$  is an upset. Hence a set is closed in the original topology iff it is closed in the order topology, so the two topologies are the same.

Conversely, we show that given a poset  $P$  with the order topology, the partial order  $\preceq$  defined above agrees with the original order  $\leq$  on  $P$ . But this is clear because for any  $x \in P$ ,  $\{y : y \geq x\} = \overline{\{x\}} = \{y : y \geq x\}$ .

Finally, we show that a map  $f : P \rightarrow Q$  between finite posets is order-preserving iff it is continuous in the order topologies. But  $f$  is continuous iff for all  $x \in Q$ ,  $f^{-1}(x^\downarrow)$  is a downset, which just says that  $f(y) \leq x$  and  $z \leq y$  imply  $f(z) \leq x$ , and it is easy to see that this is equivalent to  $f$  preserving order.  $\square$

At times, it will be convenient to consider spaces that are infinite but share most of the important properties of finite spaces. We thus make the following definition.

**Definition 1.4.** *A space is called locally finite if there exists a finite neighborhood of each point. A poset is called locally finite if the downset of each point is finite.*

**Theorem 1.5.** *The association of a locally finite poset with its order topology is an isomorphism from the category of locally finite posets to the category of locally finite spaces.*

*Proof.* The proof is almost identical to that of Theorem 1.3, and most details are left to the reader. To show that a subset  $A$  of a locally finite space  $X$  is closed in the original topology iff it is an upset in the induced ordering, note that  $A$  is closed iff there is an open covering  $X = \bigcup U_\alpha$  such that  $A \cap U_\alpha$  is closed in  $U_\alpha$  for each  $\alpha$ . We can then choose the covering of all finite open sets and apply Theorem 1.3.  $\square$

From now on, we shall make no distinction between (locally) finite spaces and (locally) finite posets. Given a finite space  $X$ , we denote the finite space given by the opposite partial order (i.e.  $x \leq^{\text{op}} y$  iff  $y \leq x$ ) by  $X^{\text{op}}$  and call it the *opposite space* of  $X$ . We make the same definition for locally finite spaces, though the opposite of a locally finite space may fail to be locally finite. If both a poset and its opposite are locally finite, we say the poset is *bilocally finite*; these will be somewhat better behaved than arbitrary locally finite spaces.

## 2 Exploring finite spaces

Most of the results in this section follow from and are greatly generalized by Theorem 3.5. However, in this section we hope to give the reader a chance to play with (locally) finite spaces and gain some intuition for their behavior before plunging into greater generality. This section also contains examples that provide motivation behind Theorem 3.5.

### 2.1 Hasse diagrams

First, we introduce a useful way of visualizing locally finite spaces. Intuitively, this consists of simply drawing the poset as a graph in the “obvious” way, with larger elements at the top of the graph. Formally, we make the following definition.

**Definition 2.1.** *Let  $X$  be a poset and  $x, y \in X$ . We say  $y$  covers  $x$  if  $x < y$  and there is no  $z \in X$  such that  $x < z < y$ . The Hasse diagram of  $X$  is the directed graph on vertex set  $X$  whose vertices are pairs  $(x, y)$  such that  $y$  covers  $x$ .*

For a locally finite poset, it is easy to see that the order relation is the transitive relation generated by the covering relation, so the poset is determined by its Hasse diagram.

**Definition 2.2.** *Let  $X$  be locally finite. Then the rank of an element  $x \in X$  is defined inductively as  $\text{rank}(x) = \max\{\text{rank}(y) + 1 : y < x\}$ , with the rank of a minimal element of  $X$  being 0. The height of  $X$  is  $\sup_{x \in X} \text{rank}(x)$  (or  $-1$  if  $X$  is empty).*

We leave it as an exercise to the reader to verify that the induction in the definition of rank is valid, and that every element of a locally finite space has finite rank. Note, however, that an infinite locally finite space (e.g.  $\mathbb{N}$  with the usual ordering) may have infinite height. If the opposite of a locally finite space is locally finite, it has the same height.

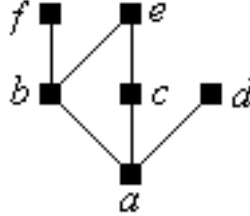
We normally draw the Hasse diagram of a locally finite poset by putting the elements of each rank in a row, with higher ranks above lower ranks. We draw the edges as undirected lines, since they must always go from lower ranks to higher ranks.

**Example 2.3.** *Consider the poset  $X = \{a, b, c, d, e, f\}$  with  $a < b$ ,  $a < c$ ,  $a < d$ ,  $a < e$ ,  $a < f$ ,  $b < e$ ,  $b < f$ , and  $c < f$ . Then the Hasse diagram of  $X$  is depicted in Figure 1.*

### 2.2 Connectedness in locally finite spaces

In this section we describe the basic (path-)connectedness properties of locally finite spaces. The key fact is the following.

Figure 1: The space in Example 2.3



**Proposition 2.4.** *Let  $X$  be a poset with either a greatest or a least element. Then  $X$  is contractible.*

*Proof.* Suppose  $a \in X$  is a greatest element. Define  $H : X \times I \rightarrow X$  by  $H(x, t) = x$  for  $t \in [0, 1/2)$  and  $H(x, t) = a$  for  $t \in [1/2, 1]$ . Then for  $x \neq a$ ,  $H^{-1}(x^\downarrow) = x^\downarrow \times [0, 1/2)$  is open, as is  $H^{-1}(a^\downarrow) = X \times I$  (since  $a^\downarrow = X$  by hypothesis). Thus  $H$  is continuous and is a homotopy from the identity on  $X$  to the constant map at  $a$ . Hence  $X$  is contractible.

The case that  $X$  has a least element is similar and left as an exercise to the reader.  $\square$

**Corollary 2.5.** *Posets (in particular, locally finite spaces) are locally contractible.*

*Proof.* Let  $X$  be a poset,  $U \subseteq X$  be open, and  $x \in U$ . Then  $x^\downarrow$  is a neighborhood of  $x$  contained in  $U$ . But  $x$  is the greatest element of  $x^\downarrow$ , so by the Proposition  $x^\downarrow$  is contractible.  $\square$

**Corollary 2.6.** *Every connected locally finite space has a universal cover, and its universal cover is locally finite of the same height.*

*Proof.* It is a standard theorem that every connected, locally path-connected, and semilocally simply connected space has a universal cover. But any locally contractible space is locally path-connected and semilocally simply connected, so any connected locally finite space has a universal cover. Since covering maps are local homeomorphisms, any covering space of a locally finite space is locally finite, and since rank is a local property, covering maps also preserve height.  $\square$

This result is one reason to be interested in locally finite spaces as well as finite spaces. As we shall see, the universal cover of a finite space may be only locally finite.

We now characterize when a locally finite space is connected.

**Definition 2.7.** *Let  $X$  be a poset. Let  $\sim$  denote the equivalence relation generated by the order relation of  $X$ . Explicitly, we say  $x \sim y$  if there exist  $z_0, \dots, z_n$  such that  $x \leq z_0 \geq z_1 \leq z_2 \geq \dots \leq z_n \geq y$ . Then we call the equivalence classes of  $\sim$  the order-components of  $X$ .*

**Proposition 2.8.** *The order-components of a poset  $X$  are exactly the (path-)connected components of the order topology on  $X$ .*

*Proof.* First, if  $A \subseteq X$  is an order-component,  $A = \bigcup_{a \in A} a^\perp$ , so  $A$  is open. But the complement of  $A$  is a union of order-complements and thus also open, so  $A$  is clopen. It follows that connected components are contained in order-components. Conversely, if  $x \leq y$ , then since  $y^\perp$  is contractible and in particular connected,  $x$  and  $y$  are in the same connected component. But being in the same connected component is an equivalence relation, so it follows that order-components are contained in connected components. Finally, the path-connected components are the same as the connected components since  $X$  is locally contractible and hence locally path-connected.  $\square$

For a locally finite space, it is easy to see that order-components coincide with connected components of the poset's Hasse diagram. Thus a locally finite space is connected iff its Hasse diagram is connected.

### 2.3 A “finite circle”

We now know when a locally finite space is connected and have a simple sufficient condition for a locally finite space to be contractible. But how, in general, might we find a locally finite space that is connected but not contractible? Is there, for example, a connected finite space that is not simply-connected? We strongly encourage the reader to explore this question for herself before reading on.

Let us try to construct the smallest possible non-contractible (nonempty) connected finite space. A connected space with 1 or 2 points clearly has a greatest element and is hence contractible. Suppose we have a connected space  $X = \{a, b, c\}$  with three points. We may assume WLOG that  $a$  is a minimal element and  $b > a$ . Then by connectedness, we must have either  $c > a$  or  $c < b$ . In the former case  $a$  is a least element and in the latter case  $b$  is a greatest element, so either way  $X$  is contractible.

Now suppose we have a connected space with 4 points. Making a similar analysis as we did for 3 points, we would find several possibly non-contractible spaces. However, instead of going through the details of that, we observe that in any locally finite space  $Y$ , if  $y \in Y$  is minimal and is covered by exactly one point of  $Y$ ,  $Y$  deformation-retracts onto  $Y - \{y\}$  by mapping  $y$  to the point that covers it (the proof is similar to that of Proposition 2.4, and the reader is encouraged to check the details). Similarly,  $Y$  also deformation-retracts onto  $Y - \{y\}$  if  $y$  is maximal and covers exactly one point. Thus since every 3-point connected space is contractible, for a 4-point space not to be contractible, each minimal point must be covered by at least two points and each maximal point must cover at least two points. Also, it must have at least two minimal points and at least two maximal points, since if there is only one minimal point it is a least point and similarly for maximal points. Thus a non-contractible connected 4-point space must have two minimal points  $a$  and  $b$  and two maximal points  $c$  and  $d$  with the Hasse diagram in Figure 2. We will (temporarily) refer to this space as  $X$ .

Figure 2: The noncontractible 4-point space  $X$

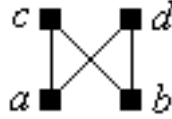
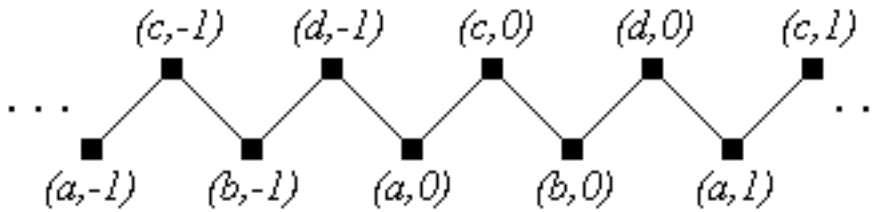


Figure 3: The universal cover  $Y = X \times \mathbb{Z}$  of  $X$



Now, it is possible to argue directly that  $X$  is not contractible (see Corollary 4.5, for example). However, in order to not only show that  $X$  is not contractible but also more closely analyze its topological properties, we construct the universal cover of  $X$ . Write  $Y = X \times \mathbb{Z}$ , with the ordering indicated in Figure 3.

It is trivial to see that the projection map  $\pi : Y \rightarrow X$  is a covering map. In particular, since  $X$  has a nontrivial cover, we conclude immediately that  $X$  is not simply connected. To show  $Y$  is the universal cover of  $X$ , we show that it is contractible. To simplify notation, we identify  $Y$  with  $\mathbb{Z}$ , ordered as shown in Figure 4.

Now for each  $N \geq 0$ , define  $f_N : Y \rightarrow Y$  by  $f_N(m) = -N$  if  $m < -N$ ,  $f_N(m) = m$  if  $m \in [-N, N]$  and  $f_N(m) = N$  if  $m > N$ . Note that each  $f_N$  is continuous and  $f_0$  is the constant map to 0. Now for each  $N$ ,  $f_N$  is homotopic to  $f_{N+1}$  because in  $[-N-1, N+1] \subseteq Y$ ,  $-N-1$  and  $N+1$  are either minimal and covered only by  $-N$  and  $N$  or maximal and only cover  $-N$  and  $N$  (depending on the parity of  $N$ ), so  $[-N-1, N+1]$  deformation-retracts to  $[-N, N]$ . We thus define  $H : Y \times I \rightarrow Y$  by letting  $H|_{Y \times [1/(N+2), 1/(N+1)]}$  be a homotopy from  $f_{N+1}$  to  $f_N$  and  $H(y, 0) = y$  for all  $y$ . Then  $H$  is clearly continuous on

Figure 4:  $Y$  identified with  $\mathbb{Z}$

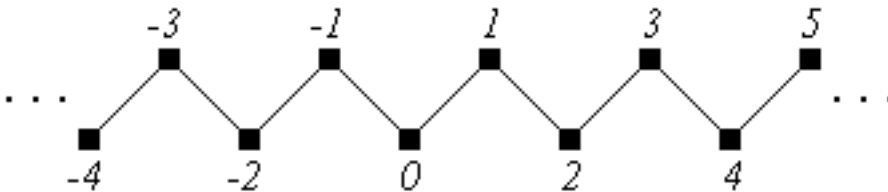
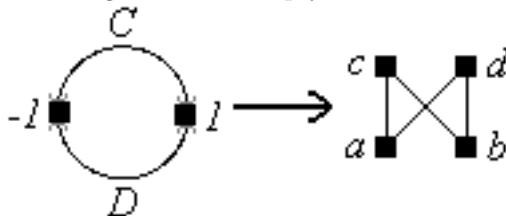




Figure 5: The map  $f : S^1 \rightarrow X$



$Y \times (0, 1]$ , and as long as we choose reasonable homotopies from  $f_{N+1}$  to  $f_N$  it is easy to see that it is also continuous at time 0. Thus  $H$  is a homotopy from the identity to the constant map  $f_0$ , so  $Y$  is contractible.

It is now difficult not to notice that  $X$  has the homotopy groups of a circle, and that the covering map  $Y \rightarrow X$  intuitively “looks like” the universal cover  $\mathbb{R} \rightarrow S^1$ . Indeed, let  $C$  be the intersection of  $S^1 \subseteq \mathbb{R}^2$  with the open upper half-plane and  $D$  be the intersection with the open lower half-plane. Then the map

$$f : \begin{cases} 1 \mapsto c \\ C \mapsto b \\ -1 \mapsto d \\ D \mapsto a \end{cases}$$

is continuous  $S^1 \rightarrow X$  and is easily seen to generate  $\pi_1(X, a_1)$  (see Figure 5).

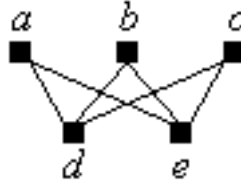
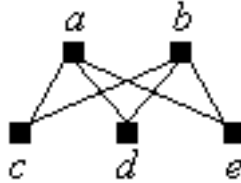
In particular, this means that  $f$  induces an isomorphism on  $\pi_1$ , and since all the other homotopy groups of  $S^1$  and  $X$  are trivial,  $f$  is a weak homotopy equivalence. We also note that  $f$  is a topological quotient map, and that we can define a similar map  $\tilde{f} : \mathbb{R} \rightarrow Y$  that commutes with the universal cover maps of  $S^1$  and  $X$ .

In light of this close relationship between  $X$  and  $S^1$ , we will from now on call  $X$  a “finite circle” and write it as  $S_f^1$ . We think of  $S_f^1$  as in some sense being a “finite model”<sup>1</sup> of the continuous space  $S^1$ .

## 2.4 Height 1 spaces

We now examine how the analysis of  $S_f^1$  above can be generalized to other finite spaces of height 1. First, consider the 5-point space in Figure 6, which we will call  $K_{32}$  because its Hasse diagram is a complete bipartite graph with 3 maximal elements and 2 minimal elements. To compute the fundamental group of this space, we can simply use van Kampen’s theorem. The subspaces  $A = \{a, b, d, e\}$  and  $B = \{b, c, d, e\}$  are each open and homeomorphic to  $S_f^1$ , and their intersection has a greatest element  $b$  and is hence contractible. By van Kampen’s

<sup>1</sup>Normally, the counterpart to “continuous” would be “discrete,” but in the case of topological spaces, “discrete” already has another meaning. For this reason we say “finite model” instead of “discrete model.”

Figure 6:  $K_{32}$ Figure 7:  $K_{23}$ 

theorem and our computation of  $\pi_1(S_f^1)$  above, we conclude that  $\pi_1(K_{32})$  is free on two generators. The Mayer-Vietoris sequence of the decomposition  $K_{32} = A \cup B$  can also be used to show that the homology groups of  $K_{32}$  are trivial in all dimensions greater than 1. We note that van Kampen's theorem and the Mayer-Vietoris sequences are valid here, as they require no separation axioms or other such niceness conditions, at least when the space is being decomposed into open subsets. It is clear that these computations can be generalized by induction to the spaces  $K_{n2}$  whose Hasse diagrams are complete bipartite graphs with  $n$  maximal elements and 2 minimal elements:  $\pi_1(K_{n2})$  is free on  $n - 1$  generators and  $H_k(K_{n2}) = 0$  for  $k > 1$ .

What happens if we have more than 2 minimal elements? Consider the simplest example of  $K_{23} = K_{32}^{\text{op}}$ , shown in Figure 7. We presumably want to split this space into two copies of  $S_f^1$ , as we did with  $K_{32}$ . However, the subspaces  $\{a, b, c, d\}$  and  $\{a, b, d, e\}$  (and also  $\{a, b, c, e\}$ ) which are homeomorphic to  $S_f^1$  are closed, not open. Thus we cannot apply van Kampen's theorem to compute the fundamental group. Indeed, it seems that the only elementary way of computing the fundamental group is to explicitly write down covering spaces. While this is fairly straightforward for  $K_{23}$ , it quickly becomes quite laborious for the more general spaces  $K_{2n} = K_{n2}^{\text{op}}$ . We spare the reader the details and simply state the final result, which is that  $\pi_1(K_{2n})$ , like  $\pi_1(K_{n2})$ , is free on  $n - 1$  generators. We note that Mayer-Vietoris still can be used to compute the homology of  $K_{2n}$  (using the subspaces  $\{a, c, d, e\}$  and  $\{b, c, d, e\}$ , for example, for  $K_{23}$ ), since it does not require the intersection of the subspaces to be connected. These straightforward computations show that  $H_k(K_{2n}) = 0$  for  $k > 1$ .

Let us now step back and try to see what is "really" going on. It is clear that the spaces  $K_{n2}$  and  $K_{2n}$  have very similar properties, even though it is easy to see that they are not homotopy equivalent for  $n > 2$  (the natural bijection

between them reverses order and is thus not continuous). This is perhaps not surprising given that, for example, the proof of Proposition 2.4 was essentially invariant replacing a space with its opposite. Nevertheless, it is not clear how one would prove in general that, say, the fundamental group of a finite space (or a bilocally finite space) is isomorphic to the fundamental group of the opposite space. One might try to use covering spaces and prove, say, that the opposite of a covering space is a covering space of the opposite space. However, to show that the opposite of a universal cover is the universal cover of the opposite space, one must show that the opposite of a simply connected space is still simply connected.

As an alternate approach, we observe that the fundamental groups and homology groups we computed for  $K_{n2}$  and  $K_{2n}$  are the same as those of the Hasse diagrams of these spaces. Indeed, the Hasse diagram of  $K_{n2}$  or  $K_{2n}$  is homotopy equivalent to a wedge of  $n - 1$  circles. One can now construct a natural quotient map from the Hasse diagram to  $K_{n2}$  or  $K_{2n}$ , similar to how we constructed a map  $S^1 \rightarrow S_f^1$  above. Using universal covers to show that the higher homotopy groups of  $K_{2n}$  and  $K_{n2}$  vanish (as we did with  $S_f^1$ , but with messier details), one can conclude that these quotient maps are in fact weak homotopy equivalences. Indeed, with more work, it is possible to show that any finite height 1 space is weak homotopy equivalent to its Hasse diagram. In particular, this implies that a finite height 1 space indeed has the same weak homotopy type as its opposite. While we invite the curious reader to explore this in more depth, we skip the details, and instead prove more general results later in Corollaries 3.7 and 3.9.

## 2.5 Finite “spheres”

We now generalize the example of  $S_f^1$  in a different direction, looking for finite models of higher-dimensional spheres. To do so, we first ask, how does one obtain actual (continuous) spheres? While there are several possible answers, probably the most natural from the perspective of algebraic topology is that spheres are suspensions of spheres of lower dimension, starting with  $S^0$ , a discrete two-point space (or  $S^{-1}$ , an empty space). Of course, we cannot take suspensions with finite spaces, since the suspension of a finite space is not finite unless the space is empty. Nevertheless, we note that  $S^0$  is already finite, and embeds as a subspace of  $S_f^1$  by mapping it to the two minimal points. Thus we can hope that  $S_f^1$  is in some sense a “suspension” of  $S^0$ , and that we can then generalize the “suspension” operation to obtain finite models of  $S^n$  for arbitrary  $n$ .

Now, the difference between  $S_f^1$  and  $S^0$  is that  $S_f^1$  contains two additional points, each of which is above the two points of  $S^0$ . Now by Proposition 2.4, adding a point that is greater than every element of a space makes the space contractible, and thus could be thought of as taking a cone on the space. Thus adding two such points is analogous to having two cones over the original space (attached at the base), which is the same as the suspension of the space. We thus define the *finite suspension* of a locally finite space  $X$  to be  $S_f X = X \sqcup \{a, b\}$ ,

ordered by the ordering on  $X$  plus  $a > x$  and  $b > x$  for all  $x \in X$ . In particular, we inductively define  $S_f^0 = S^0$  and  $S_f^{n+1} = S_f S_f^n$ , which we call “finite  $n$ -spheres.” More explicitly,  $S_f^n = \{0, 1, \dots, n\} \times \{1, -1\}$  ordered by  $(k, i) < (l, j)$  whenever  $k < l$ .

By Mayer-Vietoris and Proposition 2.4, it is easy to show that  $\tilde{H}_k(S_f X) = \tilde{H}_{k-1}(X)$  for any locally finite space  $X$ . Indeed, the proof is virtually identical to the proof of this fact for ordinary suspensions. In particular, the spaces  $S_f^n$  have the same homology as  $S^n$ . Indeed, as in the case of  $S_f^1$ , we can construct a quotient map  $S^n \rightarrow S_f^n$  that induces isomorphisms on all homology groups in the following way: given a point  $x = (x_0, x_1, \dots, x_n) \in S^n \subset \mathbb{R}^{n+1}$ , map  $x$  to  $(i, x_i/|x_i|) \in S_f^n$  for the smallest  $i$  such that  $x_i \neq 0$ . We leave it as an exercise for the reader to verify that this is a quotient map that it induces isomorphisms on homology. However, while we could use covering spaces to compute the higher homotopy groups of  $S_f^1$  and show that this quotient map is in fact a weak homotopy equivalence, for general  $S_f^n$  no such approach is available. Indeed, if the quotient map *is* a weak equivalence, then computing the homotopy groups of  $S_f^n$  is at least as hard as computing the homotopy groups of  $S^n$ . Thus in order to show that  $S_f^n$  really deserves to be called a finite sphere, we need some more direct way of relating it to  $S^n$ . This problem is nontrivial, and we solve it only with Theorem 3.5 below.

### 3 Order complexes and the Lifting Theorem

In this section we generalize the examples of the previous section to arbitrary locally finite spaces. The main result Theorem 3.5 and its corollaries show that every (locally) finite space is essentially a “(locally) finite model” of a naturally defined simplicial complex, and that conversely every simplicial complex has a “locally finite model.”

#### 3.1 Order complexes

In Section 2.5, we observed that adding a greatest point to a finite space is analogous to taking a cone over the space. We now use this to generalize the quotient maps  $S^n \rightarrow S_f^n$  constructed in that section to an arbitrary locally finite space. The formal definition below may seem messy, but the basic idea is simple: you build a “continuous model” for a finite space from the bottom up, and each time you add a point to the finite space you adjoin a cone over its downset in the continuous model.

Let  $X$  be finite, and extend the partial ordering on  $X$  to a total ordering  $\preceq$ . In particular, this means that each initial segment of the total ordering is a downset under the original ordering and hence open in  $X$ . By induction on the cardinality of  $X$ , we define a space  $\delta X$  and a quotient map  $\pi_X : \delta X \rightarrow X$  such that for each subset  $A \subseteq X$  (with the total ordering  $\preceq|_A$ ), there is a natural embedding  $\delta A \hookrightarrow \delta X$  that commutes with the quotient maps. When  $X$  is empty, we let  $\delta X$  be empty and  $\pi_X$  be the only possible map. For the induction step,

let  $x$  be the greatest element of  $X$  under the total ordering and  $Y = X - \{x\}$ . By induction we have constructed  $\delta Y$  and  $\pi_Y$ . Let  $A = Y \cap x^\perp$ . We then define  $\delta X$  to be  $\delta Y$  together with a cone on the subspace  $\delta A$ . Explicitly, we write  $\delta X = \delta Y \times \{0\} \cup \delta A \times I / \delta A \times \{1\}$ .<sup>2</sup> We then define  $\pi_X$  by  $\pi_X(y, t) = \pi_Y(y)$  for  $t \in [0, 1)$  and  $\pi_X(\delta A \times \{1\}) = x$ . We leave it to the reader to check that this is continuous and a quotient map, and that  $\delta B$  embeds naturally in  $\delta X$  for any  $B \subseteq X$ .

This construction can also be carried out for locally finite spaces if we choose  $\preceq$  to be a well-ordering and use transfinite induction, taking direct limits at limit stages. While this in general requires the axiom of choice, we shall see below that the definition of  $\delta X$  is actually completely constructive.

Now an important question is whether our construction of  $\delta X$  depended on our choice of a total ordering  $\preceq$ . That is, if we add the points to the space in a different order, do we get a different continuous model? The construction below gives a simpler description of  $\delta X$  and shows that the answer is no.

**Definition 3.1.** *Let  $X$  be a locally finite space. Then the order complex  $\Delta X$  of  $X$  is the (abstract) simplicial complex on vertex set  $X$  whose cells are given by finite nonempty chains in the partial order on  $X$ .*

**Proposition 3.2.** *Let  $X$  be a locally finite space. Then the (geometric realization of the) order complex of  $X$  is naturally homeomorphic to the space  $\delta X$  constructed above. In particular,  $\delta X$  does not depend on the choice of the ordering  $\preceq$ .*

*Proof.* For any finite nonempty chain  $C \in X$ ,  $\delta C$  naturally embeds in  $\delta X$ . We leave it to the reader to check that  $\delta C$  is in fact a simplex and that these simplices define a simplicial complex structure on  $\delta X$  that is isomorphic to the order complex of  $X$ ; the proofs of these facts are straightforward but tedious by induction.  $\square$

From now on we will make no distinction between  $\delta X$  and  $|\Delta X|$ . Chasing through the construction above, we observe that the quotient map  $\pi_X : |\Delta X| \rightarrow X$  maps the interior of a simplex  $|\Delta C|$  to the least element of the chain  $C$ . Also, for any  $A \subseteq X$ , the order complex of  $A$  is naturally a subcomplex of the order complex of  $X$ , and it is easy to see that this agrees with the previously defined embedding  $\delta A \rightarrow \delta X$ . Note that  $|\Delta A| \subseteq |\Delta X|$  is *not* the same as (or even homeomorphic to)  $\pi_X^{-1}(A)$  unless  $A$  is closed in  $X$ , but it's easy to see that  $\pi_X^{-1}(A)$  deformation-retracts onto  $|\Delta A|$ .

The association of a locally finite space with its order complex is functorial: given a map  $f : X \rightarrow Y$  of locally finite spaces,  $f$  is order-preserving and hence maps chains to chains. Thus  $f$  induces a simplicial map  $\Delta f : \Delta X \rightarrow \Delta Y$ , and it is clear that  $\Delta 1 = 1$  and  $\Delta(fg) = \Delta f \Delta g$ . Furthermore, it is easy to see that the geometric realization of such a  $\Delta f$  commutes with the quotient maps, which shows that  $\pi_X$  is a natural transformation from  $|\Delta|$  to the inclusion functor of the category of locally finite spaces into the category of all spaces.

<sup>2</sup>If  $A$  is empty, we interpret this quotient to mean that we add an isolated point to  $\delta Y \times \{0\}$ .

We note that the definition of the order complex generalizes the constructions in Sections 2.4 and 2.5. For a height 1 space, the order complex is just the Hasse diagram, since the only chains are singletons (corresponding to vertices of the Hasse diagram) and pairs with  $x < y$  (in which case  $y$  covers  $x$  and there is a corresponding edge in the Hasse diagram). Similarly, it is easy to see that  $|\Delta(S_f X)| = S(|\Delta X|)$ , and in particular  $|\Delta S_f^n| = S^n$ . As in those special cases, we may use induction and Mayer-Vietoris sequences to show that  $\pi_X$  induces isomorphisms on homology for all finite spaces  $X$ ; we encourage the reader to verify the details of this. Finally, we observe that the dimension of the simplicial complex  $\Delta X$  is the same as the height of  $X$ .

When there is no chance of ambiguity, we shall drop the subscript on the quotient map  $\pi_X$ .

### 3.2 The Lifting Theorem

We now show that the order complex of a locally finite space has a very useful universal property: any map from a “nice” space to a locally finite space factors through the order complex, uniquely up to homotopy. To define “nice” more precisely, we take a brief excursion into pointset topology.

Recall that a space  $X$  is called *perfectly normal* if it is normal and for every closed subset  $A \subseteq X$ , there is a map  $f : X \rightarrow I$  such that  $f^{-1}(\{1\}) = A$ . Any metrizable space is perfectly normal, and any subspace of a perfectly normal space is perfectly normal. If  $X$  is perfectly normal, so is the product  $X \times I$  (see [9]).

**Lemma 3.3.** *Let  $X$  be perfectly normal,  $A, B \subseteq X$  be closed, and  $f : B \rightarrow I$  be such that  $f^{-1}(\{1\}) = A \cap B$ . Then  $f$  extends to a map  $g : X \rightarrow I$  such that  $g^{-1}(\{1\}) = A$ .*

*Proof.* Let  $h : A \cup B \rightarrow I$  be given by  $h|_A = 1$  and  $h|_B = f$ . Then  $h$  is continuous, so by the Tietze extension theorem it has a continuous extension  $g_1$  to all of  $X$ . By perfect normality, let  $g_2 : X \rightarrow I$  be such that  $g_2^{-1}(\{1\}) = A \cup B$ . Then  $g = g_1 g_2$  has the desired properties.  $\square$

**Proposition 3.4.** *Any CW-complex is perfectly normal.*

*Proof.* Any CW-complex is normal, so it suffices to check the second condition. Let  $X$  be a CW-complex and  $A \subseteq X$  be closed. By (possibly transfinite) induction on the cells of  $X$ , we define a map  $f : X \rightarrow I$  with  $f^{-1}(\{1\}) = A$ . We start with the empty function on the empty subcomplex of  $X$ , and at limit stages simply glue together the previous functions.

Now we handle successor stages. If we have defined a function  $f|_B$  for some subcomplex  $B \subset X$  with  $f|_B^{-1}(\{1\}) = A \cap B$  and  $e$  is a cell of  $X - B$  whose boundary is contained in  $B$ , we can extend  $f|_B$  to  $B \cup e$  as follows. Since  $B \cup e = B \cup \bar{e}$  and both  $B$  and  $\bar{e}$  are closed, it suffices to define  $g : \bar{e} \rightarrow I$  such that  $g$  agrees with  $f|_B$  on  $\partial e$  and  $g^{-1}(\{1\}) = A \cap \bar{e}$ . Now  $\bar{e}$  intersects only finitely many cells of  $X$ , so it is metrizable and hence perfectly normal. Since  $\partial e$  and  $A \cap \bar{e}$  are closed, the existence of such a  $g$  now follows from Lemma 3.3.  $\square$

We now prove the main lifting theorem.

**Theorem 3.5** (Lifting Theorem for Finite Spaces). *Let  $X$  be perfectly normal,  $A \subseteq X$  be closed, and  $Y$  be a finite space. Then if  $f : X \rightarrow Y$  and  $\tilde{g} : A \rightarrow |\Delta Y|$  is such that  $\pi\tilde{g} = f|_A$  for  $\pi : |\Delta Y| \rightarrow Y$  the quotient map, there exists a lift  $\tilde{f} : X \rightarrow |\Delta Y|$  such that  $\pi\tilde{f} = f$  and  $\tilde{f}|_A = \tilde{g}$ . Furthermore, any two such lifts are homotopic such that every stage of the homotopy is also such a lift.*

*Proof.* First, we prove the existence of such an  $\tilde{f}$  by induction on the cardinality of  $Y$ . When  $Y$  is empty, this is trivial.

Now suppose that such a  $\tilde{f}$  always exists for any finite space of cardinality less than  $|Y|$ . Let  $y \in Y$  be any maximal point. Then there is some total ordering on  $Y$  extending the partial ordering of  $Y$  such that  $y$  is the greatest element. By Proposition 3.2, this means that for  $Z = Y - \{y\}$  and  $U = y^\perp \cap Z$ ,

$$|\Delta Y| = |\Delta Z| \times \{0\} \cup |\Delta U| \times I / |\Delta U| \times \{1\},$$

or  $|\Delta Z|$  together with a cone on  $|\Delta U|$ . Let  $V = f^{-1}(Z)$ ; then  $V$  is open in  $X$  and perfectly normal because  $X$  is. Now since  $\tilde{g}$  is a lift of  $f$ , the image of  $\tilde{g}|_{A \cap V}$  lies in  $|\Delta Y| - |\Delta(\{y\})| = |\Delta Z| \times \{0\} \cup |\Delta U| \times [0, 1)$ . We let  $g_1$  be the composition of  $\tilde{g}|_{A \cap V}$  with the projection  $|\Delta Z| \times [0, 1) \rightarrow |\Delta Z|$ ; it is easy to see that  $g_1$  is a lift of  $f|_V : V \rightarrow Z$  to  $|\Delta Z|$ . By the induction hypothesis, there is a lift  $f_0 : V \rightarrow |\Delta Z|$  of  $f|_V$  extending  $g_1$ .

Now since  $\tilde{g}$  is continuous, there is an open neighborhood  $W_0 \subseteq f^{-1}(y^\perp)$  of  $f^{-1}(\{y\})$  such that  $\bar{W}_0 \cap A \subseteq \tilde{g}^{-1}(|\Delta(y^\perp)|)$  but such that the interior of  $\tilde{g}^{-1}(|\Delta(y^\perp)|)$  is contained in  $W_0$ . Using perfect normality, one can construct a smaller open neighborhood  $W$  of  $f^{-1}(\{y\})$  containing  $W_0 \cap A$  such that  $\bar{W}$  intersects the boundary of  $W_0$  only at points of  $A$  (construct a continuous function that is 0 on  $f^{-1}(\{y\})$ ,  $< 1$  on  $W_0 \cap A$ , 1 on  $\partial W_0 \cap A$ , and  $> 1$  on the rest of  $\partial W_0$ , and take the inverse image of  $[0, 1)$ ). Since  $\pi^{-1}(y^\perp)$  deformation retracts onto  $|\Delta(y^\perp)|$  (being essentially an iterated cylinder on it), it is clear by normality that we can deform  $f_0$  on  $W_0$  to get a new lift  $f_1$  of  $f|_V$  that agrees with  $f_0$  off of  $W_0$  and such that on  $W$ , the image of  $f_1$  is contained in  $|\Delta U| = |\Delta Z| \cap |\Delta y^\perp|$ . Since the image of  $\tilde{g}$  is already contained in  $|\Delta(y^\perp)|$  on  $W_0$ , this can clearly be done such that  $f_1$  is also still an extension of  $g_1$ .

Now let  $g_2$  be the composition of  $\tilde{g} : A \rightarrow |\Delta Z| \times \{0\} \cup |\Delta U| \times I / |\Delta U| \times \{1\}$  with the projection  $|\Delta Z| \times I / |\Delta Z| \times \{1\} \rightarrow I$  onto the second coordinate (which is still well-defined and continuous even for the quotient). Let  $C = X - W$  and  $D = f^{-1}(\{y\})$ ; these are closed and disjoint in  $X$ . Define  $h : A \cup C \rightarrow I$  by  $h|_A = g_2$  and  $h|_C = 0$ . The condition that  $\tilde{g}$  is a lift of  $f$  implies that this is well-defined and hence continuous since  $A$  and  $C$  are closed. Also, since  $\tilde{g}$  is a lift of  $f$ ,  $h^{-1}(\{1\}) = A \cap D$ . By Lemma 3.3, let  $f_2 : X \rightarrow I$  extend  $h$  such that  $f_2^{-1}(\{1\}) = D$ .

Now define  $\tilde{f}(x) = (f_1(x), f_2(x)) \in |\Delta Z| \times \{0\} \cup |\Delta U| \times I / |\Delta U| \times \{1\}$  for  $x \in V$  and  $\tilde{f}(x) = |\Delta U| \times \{1\}$  (i.e., the point  $|\Delta(\{y\})|$ ) for  $x \in D = X - V$ . Since  $f_2 = 0$  on  $C$ ,  $x \in V$  and  $f_2(x) > 0$  imply  $f_1(x) \in |\Delta U|$  by definition of  $C$ , so this is indeed well-defined. Since  $f_2^{-1}(\{1\}) = D$  this is continuous, and it is easy to check that it is a lift of  $f$  extending  $\tilde{g}$ .

Now suppose  $\tilde{f}_0$  and  $\tilde{f}_1$  are two such lifts. Let  $H : X \times I \rightarrow Y$  be given by  $H(x, t) = f(x)$ . Define  $G : X \times \{0, 1\} \cup A \times I \rightarrow |\Delta Y|$  by  $G(x, 0) = \tilde{f}_0(x)$ ,  $G(x, 1) = \tilde{f}_1(x)$ , and  $G(x, t) = \tilde{g}(x)$  for  $x \in A$ . Then  $G$  is continuous and a lift of  $H$  since the  $\tilde{f}_i$  are lifts of  $f$  extending  $\tilde{g}$ . By the first part of the theorem, then, there is a lift  $\tilde{H}$  of  $H$  to  $|\Delta Y|$  that extends  $G$ , and such a lift is a homotopy from  $\tilde{f}_0$  to  $\tilde{f}_1$ , every stage of which is a lift of  $f$  extending  $\tilde{g}$ .  $\square$

Although this theorem only directly applies to finite spaces, the following fact allows us to use it on arbitrary locally finite spaces.

**Lemma 3.6.** *A locally finite space is compact iff it is finite.*

*Proof.* Clearly any finite space is compact. Conversely, the set of all finite open subsets of an infinite locally finite space has no finite subcover.  $\square$

We can now prove the most important corollary of the lifting theorem for arbitrary locally finite spaces.

**Corollary 3.7.** *Let  $X$  be a locally finite space. Then the quotient map  $\pi : |\Delta X| \rightarrow X$  is a weak homotopy equivalence.*

*Proof.* Fix any basepoint  $x \in |\Delta X|$ . Let  $f : (S^n, p) \rightarrow (X, \pi(x))$  be a basepointed map. By Lemma 3.6,  $Y = f(S^n)$  is finite. By Theorem 3.5 (with  $A = \{\pi(x)\}$ ), there is a basepointed lift  $\tilde{f} : (S^n, p) \rightarrow (|\Delta Y|, x)$  of  $f : S^n \rightarrow Y$ . Since the inclusion  $|\Delta Y| \hookrightarrow |\Delta X|$  commutes with the quotient maps,  $\tilde{f} : (S^n, p) \rightarrow (|\Delta X|, x)$  is also a lift of  $f : (S^n, p) \rightarrow (X, \pi(x))$ . This shows that the induced map  $\pi_* : \pi_n(|\Delta X|, x) \rightarrow \pi_n(X, \pi(x))$  on homotopy groups is surjective. For injectivity, suppose  $f_0, f_1 : (S^n, p) \rightarrow (|\Delta X|, x)$  and  $H : S^n \times I \rightarrow X$  is a basepointed homotopy from  $\pi f_0$  to  $\pi f_1$ . Let  $G : S^n \times \{0, 1\} \cup \{p\} \times I \rightarrow |\Delta X|$  be given by  $G(x, 0) = f_0(x)$ ,  $G(x, 1) = f_1(x)$ , and  $G(p, t) = x$ . Then as above there is a lift  $\tilde{H}$  of  $H$  to  $|\Delta X|$  that extends  $G$ , which defines a basepointed homotopy from  $f_0$  to  $f_1$ .  $\square$

By the Whitehead theorem, it follows that  $\pi$  induces isomorphisms on homology, though we could also prove this directly by a similar lifting argument.

This weak equivalence also has a functorial aspect:

**Corollary 3.8.** *Let  $f : X \rightarrow Y$  be a map between locally finite spaces. Then the induced map  $|\Delta f| : |\Delta X| \rightarrow |\Delta Y|$  induces the same maps on homotopy and (co)homology groups as  $f$ , when we identify the homotopy and (co)homology groups of  $X$  and  $Y$  with those of  $|\Delta X|$  and  $|\Delta Y|$  via  $\pi_X$  and  $\pi_Y$ .*

*Proof.* This is immediate from the fact that  $|\Delta f|$  commutes with the quotient maps (i.e., the naturality of the quotient maps).  $\square$

In the special case that  $f : Y \hookrightarrow X$  is an inclusion, this implies by the five lemma that  $\pi$  induces isomorphisms on relative homotopy and (co)homology groups from  $(|\Delta Y|, |\Delta X|)$  to  $(Y, X)$ .

As another corollary, we obtain an answer to a question raised in Section 2.4.



**Corollary 3.9.** *Let  $X$  be a bilocally finite space. Then  $X$  and  $X^{\text{op}}$  are weak homotopy equivalent.*

*Proof.* It is clear that  $X$  and  $X^{\text{op}}$  have the same order complex, so this follows from the previous corollary.  $\square$

We now record a version of the lifting theorem for locally finite spaces.

**Corollary 3.10** (Lifting Theorem for Locally Finite Spaces). *Use the same notation and hypotheses as in Theorem 3.5, except assume  $Y$  is only locally finite. Then if  $X$  is either compact or a CW-complex, the conclusion of Theorem 3.5 still holds.*

*Proof.* If  $Y$  is compact, the image of  $f$  is finite, so we proceed as in the proof of Corollary 3.7. If  $X$  is a CW-complex, we define  $\tilde{f}$  by induction on the cells of  $X$ . Given a lift  $\tilde{f}|_B$  for a subcomplex  $B \subset X$  and a cell  $e$  whose boundary is contained in  $B$ , we can use Theorem 3.5 to extend  $\tilde{f}|_{\partial e}$  to all of  $\bar{e}$  since  $\bar{e}$  is compact, and thus we obtain an extension of  $\tilde{f}$  to  $B \cup e$ . Thus we can construct an  $\tilde{f}$  cell-by-cell by induction. For uniqueness up to homotopy, use the same argument as in the proof of Theorem 3.5, noting that  $X \times I$  is also a CW-complex.  $\square$

For a discussion of further generalizations of Theorem 3.5, see Section 5.1.

Finally, we show that the “continuous model” of a locally finite space is unique up to homotopy equivalence.

**Corollary 3.11.** *Let  $Y$  be a locally finite space,  $DY$  be a CW-complex, and  $p : DY \rightarrow Y$  be a map. Then if Corollary 3.10 holds with  $|\Delta Y|$  replaced by  $DY$  and  $\pi$  replaced by  $p$ , there is a homotopy equivalence  $g : |\Delta Y| \rightarrow DY$  with homotopy inverse  $h$  such that  $pg = \pi$  and  $\pi h = p$ , and  $g$  and  $h$  are unique in having this property up to homotopy. Furthermore the homotopies  $gh \simeq 1$  and  $hg \simeq 1$  may be chosen such that each stage is a lift of  $p$  or  $\pi$ , respectively.*

*Proof.* By lifting  $\pi$  to  $DY$  and  $p$  to  $|\Delta Y|$ , we obtain maps  $g : |\Delta Y| \rightarrow DY$  and  $h : DY \rightarrow |\Delta Y|$  such that  $pg = \pi$  and  $\pi h = p$ , and these maps are unique up to homotopy. Furthermore,  $gh$  and  $hg$  are then lifts of  $p$  and  $\pi$  to  $DY$  and  $|\Delta Y|$ , respectively. Thus by the uniqueness of lifts up to homotopy,  $gh$  and  $hg$  are homotopic to the identity through such lifts.  $\square$

By essentially the same argument, one could prove a similar uniqueness property when  $DY$  satisfies the lifting theorem only up to homotopy (that is, in the conclusion of Theorem 3.5, we only require  $\pi\tilde{f}$  to be homotopic to  $f$ ).

### 3.3 Barycentric subdivision and order complexes

In this section we prove a sort of converse to Corollary 3.7, showing that every space is weak homotopic to a locally finite space.

First, let  $X$  be any abstract simplicial complex. We identify  $X$  not with its set of cells, but with its underlying vertex set, and write  $F(X)$  for the set of

cells of  $X$ . Recall that we may geometrically define the (*barycentric*) *subdivision* of  $|X|$  by adding a vertex at the barycenter of each cell; the subdivision is then a simplicial complex whose geometric realization is naturally homeomorphic to  $|X|$ . It is well-known that the subdivision has vertex set  $F(X)$  and cells corresponding to chains of cells of  $X$  ordered by inclusion. That is, if we order  $F(X)$  by inclusion (which is obviously locally finite), the subdivision of  $X$  is  $\Delta(F(X))$ , which we shall abbreviate as  $\Delta_F X$ . We note that a simplicial map  $X \rightarrow Y$  induces an order-preserving map  $F(X) \rightarrow F(Y)$ , so  $F$  is a functor from simplicial complexes to locally finite spaces.

As an immediate consequence, we have the following.

**Theorem 3.12.** *Every simplicial complex is weak homotopy equivalent to a locally finite space.*

*Proof.* Let  $X$  be an abstract simplicial complex. Then by Corollary 3.7,  $|\Delta_F X|$  is weak homotopy equivalent to  $F(X)$ , considered as a locally finite space as above. But  $|\Delta_F X|$  is homeomorphic to  $|X|$ , so we conclude that  $|X|$  is weak homotopy equivalent to  $F(X)$ .  $\square$

**Corollary 3.13.** *Every topological space is weak homotopy equivalent to a locally finite space.*

*Proof.* Every topological space is weak homotopy equivalent to a simplicial complex (see, for example, Theorem 2C.5 and Proposition 4.13 of [6]).  $\square$

Now suppose  $X$  is any locally finite space. By the remarks above, this includes the case of (the cell poset of) a simplicial complex. Since  $\Delta X$  is a simplicial complex, we may consider the locally finite space  $F(\Delta X)$ . The “continuous model” of  $F(\Delta X)$  is then the geometric realization  $|\Delta_F \Delta X|$ . By the remarks above,  $|\Delta_F \Delta X|$  is homeomorphic to (in fact, the subdivision of)  $|\Delta X|$ , so the locally finite spaces  $F(\Delta X)$  and  $X$  are weak homotopy equivalent. It is natural to wonder whether there is actually a natural weak homotopy equivalence directly from  $F(\Delta X)$  to  $X$ . To put it another way, we are looking for a map  $F(\Delta X) \rightarrow X$  that is analogous to the quotient map  $\pi : |\Delta X| \rightarrow X$ . We can make the word “analogous” precise, since there is a natural function  $q : |\Delta X| \rightarrow F(\Delta X)$  sending a point in the interior of a cell  $|\Delta C|$  to the vertex  $C \in F(\Delta X)$ : we want a naturally defined map  $p : F(\Delta X) \rightarrow X$  such that  $\pi = pq$ .

We now recall that the map  $\pi : |\Delta X| \rightarrow X$  sends the interior of a cell  $|\Delta C|$  to the least element of the chain  $C$  in  $X$ . Thus if we let  $p$  be the map taking a chain  $C$  to its least element, we have  $\pi = pq$ . However, this map is not in general continuous; indeed, it is order-reversing, not order-preserving. To make it order-preserving, we must reverse the ordering on  $F(\Delta X)$ . Thus from now on we will consider  $F(\Delta X)$  to be ordered by *reverse inclusion*, not inclusion. We also note that in general, the map  $q : |\Delta X| \rightarrow F(\Delta X)$  defined above is continuous (and in fact a quotient map) only when  $\Delta X$  is ordered by reverse inclusion.

Thus it would appear that the natural ordering to put on  $F(\Delta X)$  would be reverse inclusion of cells, rather than inclusion. However, there is a technical problem with this: the ordering by reverse inclusion is not always locally finite. Indeed, it is easy to see that  $\Delta X$  is locally finite under this ordering iff  $X$  is bilocally finite, and in that case  $F(\Delta X)$  is also bilocally finite. Thus the category of bilocally finite spaces is closed under “subdivision,” but the category of all locally finite spaces is not. Also, bilocally finite spaces have another fairly natural description: they are exactly the order spaces whose order complexes are “locally finite” in the sense that every point has a neighborhood that intersects only finitely many cells. However, it is easy to see that every component of a bilocally finite space must be countable and in particular connected bilocally finite spaces have countably generated homology, so bilocally finite spaces fail to satisfy Theorem 3.12 and Corollary 3.13. These technical difficulties with infinite locally finite spaces will be revisited in Section 5.1.

We summarize the above discussion in the following proposition.

**Proposition 3.14.** *The association  $X \mapsto F(\Delta X)$  is a functor from the category of bilocally finite spaces to itself. The transformation  $p : F(\Delta X) \rightarrow X$  sending a chain to its least element is natural and a weak homotopy equivalence.*

*Proof.* We noted above that  $\Delta$  is functorial from locally finite spaces to simplicial complexes. Given a map  $f : X \rightarrow Y$ , there is thus an induced simplicial map  $\Delta f : \Delta X \rightarrow \Delta Y$  which clearly induces an order-preserving map  $F(\Delta f) : F(\Delta X) \rightarrow F(\Delta Y)$ . It follows that  $F(\Delta)$  is a functor. Naturality of  $p$  is then trivial, and the rest of the Proposition is contained in the discussion above.  $\square$

Since we consider  $F(\Delta X)$  to be ordered by reverse inclusion, we will also find it more natural to order the poset of cells of any simplicial complex by reverse inclusion, rather than by inclusion as originally indicated above. Note that for the locally finite space  $F(X)$ , the map  $p$  above goes from  $F(\Delta_F X)$  to  $F(X)$  and corresponds to a simplicial map  $\Delta_F X \rightarrow X$ . This map sends a cell of the barycentric subdivision to the cell of the original complex in which it is contained. Thus from now on, we always consider the set of cells  $F(X)$  of a simplicial complex to be ordered by reverse inclusion. As above, this ordering is not in general locally finite, but we will be most interested in it for finite simplicial complexes.

It is worth noting that the process of taking order complexes is essentially the same as taking subdivisions, and thus it would be natural to write  $\Delta$  for both the order complex functor of finite spaces and the subdivision functor  $\Delta_F$  of simplicial complexes. By extension, we could also not distinguish between  $X$  and  $F(X)$  for a simplicial complex  $X$ . While this notation would be more compact and perhaps more natural, it could also lead to confusion. Thus we will continue to write “ $F$ ” and make a clear distinction between simplicial complexes and posets. However, we *will* refer to the space  $F(\Delta X)$  as the “subdivision” of a (bilocally) finite space  $X$ . Thus we have a functor  $\Delta$  from posets to complexes and a functor  $F$  from complexes to posets, and we call their composition in either order “subdivision.”

## 4 Maps between (locally) finite spaces

By the results of the previous section, locally finite spaces are essentially the same as simplicial complexes with respect to weak homotopy. However, if we insist on actual homotopy, locally finite spaces seem to be much more complicated. In this section we prove some basic facts about maps between locally finite spaces and homotopies between them.

When dealing with maps whose domain is locally finite, there does seem to be a significant difference between finite spaces and infinite locally finite spaces. For this reason, most of the results in this section will only hold when the domains of the maps involved are actually finite.

### 4.1 Spaces of maps

We begin by analyzing spaces of maps between order spaces to obtain a combinatorial characterization of when maps on a finite space are homotopic.

Recall that in Section 2, whenever we showed that a finite space was contractible, we did so by contracting it in a series of discrete steps, such as removing a minimal point that is covered by only one other point. We now generalize this notion in the obvious way.

**Definition 4.1.** *Let  $X$  and  $Y$  be spaces. Then a map  $H : X \times I \rightarrow Y$  is an up-homotopy of maps from  $X$  to  $Y$  if that  $H(x, t) = H(x, 0)$  for all  $x \in X$  and all  $t \in [0, 1)$  and a down-homotopy if  $H(x, t) = H(x, 1)$  for all  $x \in X$  and all  $t \in (0, 1]$ . A step-homotopy is either an up-homotopy or a down-homotopy, and a finite-homotopy is a composition of (finitely many) step-homotopies.*

That is, two maps  $f$  and  $g$  are step-homotopic if you can continuously “jump” from  $f$  to  $g$  with nothing in between and finite-homotopic if you can get from  $f$  to  $g$  by a finite sequence of such jumps. The reason for the terms “up-homotopy” and “down-homotopy” is the following fact.

**Proposition 4.2.** *Let  $f, g : X \rightarrow Y$  be maps from a space  $X$  to a poset  $Y$ . Then  $f$  is up-homotopic to  $g$  iff  $f(x) \leq g(x)$  for all  $x \in X$  and down-homotopic to  $g$  iff  $f(x) \geq g(x)$  for all  $x \in X$ .*

*Proof.* By definition,  $f$  is up-homotopic to  $g$  iff the map  $H$  given by  $H(x, t) = f(x)$  for  $t \in [0, 1)$  and  $H(x, 1) = g(x)$  is continuous. For any  $y \in Y$ ,  $H^{-1}(y^\downarrow) = f^{-1}(y^\downarrow) \times [0, 1) \cup g^{-1}(y^\downarrow) \times \{1\}$ , and it is easy to see that this is open iff  $g(x) \leq y$  implies  $f(x) \leq y$  for all  $x$ . Thus  $H$  is continuous iff  $g(x) \leq y$  implies  $f(x) \leq y$  for all  $x$  and  $y$ , which is easily seen to be equivalent to  $f \leq g$  pointwise. The case of down-homotopy is similar.  $\square$

We now note that for any space  $X$  and a poset  $Y$ , the set  $Y^X$  of functions from  $X$  to  $Y$  is naturally partially ordered by taking the partial order on  $Y$  pointwise. In particular, by restriction the set  $C(X, Y)$  of continuous maps from  $X$  to  $Y$  is partially ordered. The result above together with Proposition 2.8 then says that two maps are finite-homotopic iff they are in the same component of

the order topology on  $C(X, Y)$ . We note that if  $X$  is finite and  $Y$  is locally finite,  $C(X, Y)$  is locally finite.

Now there is also another natural topology on  $C(X, Y)$ , the compact-open topology. Recall that for any spaces  $X$  and  $Y$ , the compact-open topology on the set  $C(X, Y)$  of continuous maps from  $X$  to  $Y$  is the topology generated by the sets  $D(K, U) = \{f : f(K) \subseteq U\}$ , where  $K$  ranges over all compact subsets of  $X$  and  $U$  ranges over all open subsets of  $Y$ . The following theorem, whose proof we include because it is sometimes stated only for locally compact Hausdorff spaces, shows that this topology is the “right” topology for spaces of maps between sufficiently well-behaved spaces.

**Theorem 4.3.** *Let  $X$  be a locally compact space, in the sense that every open set contains a compact neighborhood of each point. Then for any spaces  $Y$  and  $Z$ , a map  $f : X \times Z \rightarrow Y$  is continuous iff the map  $\hat{f} : Z \rightarrow C(X, Y)$  given by  $\hat{f}(z)(x) = f(x, z)$  is well-defined and continuous for the compact-open topology.*

*Proof.* Suppose  $f$  is continuous. Then  $\hat{f}(z)$  is continuous for all  $z$ , so  $\hat{f}$  is well-defined. Now suppose  $K \subseteq X$  is compact and  $U \subseteq Y$  is open. We want to show that  $A = \hat{f}^{-1}(D(K, U))$  is open; let  $z \in A$ . Then  $f(K \times \{z\}) \subseteq U$ , so since  $f$  is continuous and  $K$  is compact, there are open neighborhoods  $V$  and  $W$  of  $K$  and  $z$  respectively such that  $f(V \times W) \subseteq U$ . But then  $W \subseteq A$  and  $z \in W$ . Hence  $A$  is open, so  $\hat{f}$  is continuous.

Conversely, suppose  $\hat{f}$  is well-defined and continuous. Let  $U \subseteq Y$  be open and suppose  $f(x, z) \in U$ . Then  $g = \hat{f}(z)$  is continuous, so  $g^{-1}(U) \subseteq X$  is a neighborhood of  $x$ . By local compactness, let  $K \subseteq g^{-1}(U)$  be a compact neighborhood of  $x$ . Then  $V = \hat{f}^{-1}(D(K, U)) \subseteq Z$  is a neighborhood of  $z$ . But then  $W = K \times V$  is a neighborhood of  $(x, z)$  such that  $f(W) \subseteq U$ . Hence  $f$  is continuous.  $\square$

In particular, in the case  $Z = C(X, Y)$  we find that the evaluation map  $e : X \times C(X, Y) \rightarrow Y$  given by  $e(x, f) = f(x)$  is continuous because  $\hat{e} : C(X, Y) \rightarrow C(X, Y)$  is the identity.

We now compare the compact-open topology on  $C(X, Y)$  with the order topology when  $Y$  is a poset and  $X$  is locally finite.

**Proposition 4.4.** *Let  $X$  be a locally finite space and  $Y$  be a poset. Then the order topology on  $C(X, Y)$  is finer than the compact-open topology. If  $X$  is finite, the order topology is the same as the compact-open topology.*

*Proof.* Let  $K \subseteq X$  be compact and  $U \subseteq Y$  be open. By Lemma 3.6,  $K$  is finite, so since  $D(K, U) = \bigcap_{x \in K} D(\{x\}, U)$ , we may assume  $K = \{x\}$  is a singleton. Then  $D(\{x\}, U) = \{f \in C(X, Y) : f(x) \in U\}$  is clearly a downset since  $U$  is, and hence it is open in the order topology. Thus the order topology is finer than the compact-open topology.

Now suppose  $X$  is finite and let  $f \in C(X, Y)$ . Then  $f^\perp = \bigcap_{x \in X} \{g : g(x) \leq f(x)\} = \bigcap D(\{x\}, f(x)^\perp)$  is open in the compact-open topology. Hence the compact-open topology is finer than the order topology.  $\square$

**Corollary 4.5.** *Continuous maps from a finite space to a poset are homotopic iff they are finite-homotopic.*

*Proof.* Obviously step-homotopic maps are always homotopic. Now suppose  $X$  is finite,  $Y$  is a poset,  $f, g : X \rightarrow Y$  are continuous and  $H : X \times I \rightarrow Y$  is a homotopy from  $f$  to  $g$ . Clearly, any locally finite space is locally compact in the sense of Theorem 4.3, so  $\hat{H} : I \rightarrow C(X, Y)$  is a path from  $f$  to  $g$  and is continuous in the compact-open topology. By Proposition 4.4,  $\hat{H}$  is also continuous in the order topology. In particular,  $f$  and  $g$  are in the same component of  $C(X, Y)$  in the order topology and hence they are finite-homotopic by the discussion following Proposition 4.2.  $\square$

Corollary 4.5 is not true in general for infinite locally finite spaces. For example, in Section 2.3 we showed that the universal cover of  $S_f^1$  is contractible via an infinite “composition” of step-homotopies, and it is easy to check that it is impossible to do so with only finitely many (in fact, no map is step-homotopic to the identity other than the identity itself). It seems unlikely that there is any useful result analogous to Corollary 4.5 that allows infinitely many steps.

## 4.2 Strong homotopy classes of finite spaces

Using Corollary 4.5, we now show that finite spaces are actually quite rigid under homotopy. The material in this section is taken from [12]. Recall that in Section 2.3, we used deformation retractions that removed maximal points that covered exactly one point or minimal points that were covered by exactly one point. We now show that this is essentially the only kind of deformation retraction that exists for finite spaces. First, note that if  $y$  is the only point that covers (or is covered by)  $x$ , the identity is up-homotopic (down-homotopic) to the map sending  $x$  to  $y$  ( $y$  to  $x$ ) and which is otherwise the identity. Here is a sort of converse to this fact.

**Lemma 4.6.** *Let  $X$  be a finite space and suppose the identity map on  $X$  is up-homotopic (or down-homotopic) to a map  $f : X \rightarrow X$ . Then there exists a point  $x \in X$  such that  $f(x)$  is the only element of  $X$  that covers  $x$  (is covered by  $x$ ).*

*Proof.* By replacing  $X$  with  $X^{\text{op}}$ , it suffices to consider the up-homotopy case. Let  $x \in X$  be a minimal element such that  $f(x) > x$ . Then since  $g$  is continuous, for all  $y > x$ ,  $f(x) \leq f(y) = y$ . Thus in particular, there is no  $y > x$  with  $y < f(x)$ , so  $f(x)$  covers  $x$ , and for any  $y > x$  other than  $f(x)$ ,  $x < f(x) < y$ , so  $y$  does not cover  $x$ . Hence  $x$  has the desired property.  $\square$

**Definition 4.7.** *A unique cover in a poset  $X$  is a pair of points  $x, y \in X$  such that either  $y$  is the only point covered by  $x$  or  $y$  is the only point covering  $x$ . We call a space rigid if the identity map on the space is homotopic only to itself.*

**Corollary 4.8.** *A finite space is rigid iff it has no unique covers.*

*Proof.* By Corollary 4.5 and Lemma 4.6, a space with no unique covers is rigid. By the remarks preceding Lemma 4.6, if  $x, y \in X$  is a unique cover, then  $X$  deformation-retracts to  $X - \{x\}$  (or  $X - \{y\}$ ).  $\square$

We now show that rigid spaces correspond exactly to the (strong) homotopy types of finite spaces.

**Theorem 4.9.** *Let  $X$  be a finite space. Then there is a unique rigid space  $Y$ , the rigidification of  $X$  such that there is a quotient map  $p : X \rightarrow Y$  that is a homotopy equivalence. Furthermore,  $Y$  is also a deformation retract of  $X$  and is the only rigid space that is homotopy equivalent to  $X$ .*

*Proof.* Set  $X = X_0$ . If  $x, y \in X_n$  are a unique cover, inductively define  $X_{n+1}$  to be the quotient of  $X_n$  that identifies  $x$  and  $y$ . Note that the fact that  $x$  covers or is covered by  $y$  implies that this quotient is still  $T_0$ . Since  $X$  is finite, eventually we will reach some  $X_n$  that has no covers and which is thus rigid. We set  $Y = X_n$  and define  $p$  as the composition of the quotient maps  $X_0 \xrightarrow{p_0} X_1 \xrightarrow{p_1} \dots \xrightarrow{p_{n-1}} X_n$ . For each  $i < n$ , let  $q_i$  be either of the two left inverses to  $p_i$  (i.e., in the notation above,  $q_i$  maps the equivalence class  $x, y \in X_{i+1}$  to either  $x$  or  $y$  and otherwise is the identity). Then  $q_i$  is clearly an embedding, with image  $X_i - \{x\}$  or  $X_i - \{y\}$ . By the remarks preceding Lemma 4.6, this image is a deformation retract of  $X_i$ , and it is clear that  $p_i$  is the corresponding homotopy inverse to  $q_i$ .

Thus letting  $q$  be the composition of the  $q_i$ ,  $q : Y \rightarrow X$  is an embedding whose image is a deformation retract of  $X$ , and the corresponding homotopy inverse is  $p$ . Hence  $p$  is a homotopy equivalence and  $Y$  is a deformation retract of  $X$ . It is immediate from the definition that a homotopy equivalence between two rigid spaces must be a homeomorphism, so any other rigid space homotopic to  $X$  must be homeomorphic to  $Y$ .  $\square$

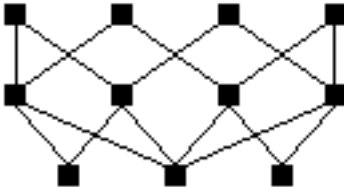
**Corollary 4.10.** *Two finite spaces are homotopy equivalent iff they have the same rigidification.*

By the construction of the rigidification in the proof of Theorem 4.9, Corollary 4.10 says two finite spaces are homotopy equivalent iff you can turn one into the other by first collapsing some unique covers one at a time and then adding some unique covers one at a time.

### 4.3 Simplicial approximation and finite spaces

While by Theorem 3.5, a finite space has the same homotopy classes of maps from nice spaces as its order complex, the situation for maps from finite spaces to other spaces is very different. First, any map from a finite space (indeed, any order space) to a  $T_1$  space must be constant on components; we leave the proof of this to the reader. Furthermore, even when the codomain is another (locally) finite space, there are much fewer maps from a finite space than from its order complex. For example, there is a homotopy class of maps  $S^1 \rightarrow S_f^1$  for each winding number in  $\mathbb{Z}$ , but using Corollary 4.5 it is easy to see that only the

Figure 8: A rigid space whose order complex is contractible



winding numbers  $-1$ ,  $0$ , and  $1$  can be realized by maps  $S_f^1 \rightarrow S_f^1$ . Similarly, while the canonical map  $p : F(\Delta S_f^1) \rightarrow S_f^1$  is a weak homotopy equivalence, and the induced map on order complexes is a homeomorphism, any map  $S_f^1 \rightarrow F(\Delta S_f^1)$  is nullhomotopic.

One can find even worse counterexamples than the ones above. For instance, by Corollary 4.8, a space whose order complex is contractible but which has no unique covers is rigid and hence not itself contractible (unless it is already a point). An example of such a space is depicted in Figure 8, originally from [3]. One can simply draw the order complex of this space and see that it is homeomorphic to a disk, even though the finite space is rigid.

However, if we allow ourselves to first subdivide our finite spaces, it is possible to realize every map on the order complex as a map on the finite space. This result is basically an adaptation of the classical simplicial approximation theorem, which we state below.

**Theorem 4.11.** *Let  $X$  and  $Y$  be finite simplicial complexes and let  $f : |X| \rightarrow |Y|$ . Then for some  $n$ ,  $f$  is homotopic to a simplicial map  $|\Delta_F^n X| \rightarrow |Y|$ , where  $\Delta_F^n X$  denotes the  $n$ th subdivision of  $X$ .*

*Proof.* See, for example, Theorem 2C.1 of [6] or Theorem 3.4.8 of [10].  $\square$

Now if  $X$  and  $Y$  are finite spaces and we have a map  $f : |\Delta X| \rightarrow |\Delta Y|$ , we can consider the  $n$ th “subdivision”  $F(\Delta_F^{n-1} \Delta X)$  of  $X$  as a finite space. We want to use Theorem 4.11 to obtain a map  $\hat{f} : F(\Delta_F^{n-1} \Delta X) \rightarrow Y$  such that the induced map on order complexes is homotopic to  $f$ . The theorem gives us a simplicial map  $h : \Delta_F^{n-1} \Delta X \rightarrow \Delta Y$ , which gives a function  $g : F(\Delta_F^{n-2} \Delta X) \rightarrow Y$  on the vertex sets of these complexes. However, this  $g$  need not be continuous as a map of finite spaces: the condition that  $h$  be simplicial says only that  $g$  maps chains to chains, which does not in general imply that  $g$  is order-preserving. However, if we subdivide  $g$  to get  $F(\Delta g) : F(\Delta_F^{n-1} \Delta X) \rightarrow F(\Delta Y)$  then  $F(\Delta g) = F(h)$  will always be order preserving, since  $F$  is functorial. We can then compose  $F(h)$  with the canonical map  $p : F(\Delta Y) \rightarrow Y$  to obtain a continuous map  $\hat{f} : F(\Delta_F^{n-1} \Delta X) \rightarrow Y$ . Since by hypothesis (the geometric realization of)  $h$  is homotopic to  $f$  and the geometric realization of  $p$  is the identity (i.e., the natural homeomorphism  $|\Delta_F \Delta Y| \rightarrow |\Delta Y|$ ), we conclude that the geometric realization of  $\hat{f}$  is homotopic to  $f$ . Thus we have the following theorem.



**Theorem 4.12.** *Let  $f : |\Delta X| \rightarrow |\Delta Y|$  for a finite space  $X$  and a locally finite space  $Y$ . Then for some  $n$ , there is a map  $\hat{f} : F(\Delta_F^{n-1}\Delta X) \rightarrow Y$  such that  $|\Delta \hat{f}|$  is homotopic to  $f$ .*

*Proof.* The only part not contained in the discussion above is the possibility that  $Y$  is only locally finite rather than finite. But since  $X$  is finite,  $|\Delta X|$  is compact and hence the image of  $f$  is contained in a finite subcomplex of  $|\Delta Y|$ , which is easily seen to be contained in  $|\Delta Z|$  for some finite subspace  $Z \subseteq Y$ . The discussion above then applies with  $Z$  in the place of  $Y$ , which proves the theorem.  $\square$

By the lifting theorem, this theorem also applies to maps  $f$  whose codomain is  $Y$  itself:

**Corollary 4.13.** *Let  $f : |\Delta X| \rightarrow Y$  for a finite space  $X$  and a locally finite space  $Y$ . Then for some  $n$ , there is a map  $\hat{f} : F(\Delta_F^{n-1}\Delta X) \rightarrow Y$  such that for  $\pi_X : |\Delta X| = |\Delta_F^n \Delta X| \rightarrow F(\Delta_F^{n-1}\Delta X)$  the quotient map,  $\hat{f}\pi_X$  is homotopic to  $f$ .*

*Proof.* By Corollary 3.10, let  $\tilde{f} : |\Delta X| \rightarrow |\Delta Y|$  be such that  $\pi_Y \tilde{f} = f$ . By Theorem 4.12, for some  $n$  there is a map  $\hat{f} : F(\Delta_F^{n-1}\Delta X) \rightarrow Y$  such that  $|\Delta \hat{f}|$  is homotopic to  $\tilde{f}$ . But  $\pi_Y |\Delta \hat{f}| = \hat{f}\pi_X$  by the naturality of  $\pi$ , so this implies  $\hat{f}\pi_X$  is homotopic to  $\pi_Y \tilde{f} = f$ .  $\square$

Another way of looking at this result is by considering the following infinite commutative diagram:

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & |\Delta_F^2 \Delta X| & \longrightarrow & |\Delta_F \Delta X| & \longrightarrow & |\Delta X| \\
& & \downarrow \pi_2 & & \downarrow \pi_1 & & \downarrow \pi_0 \\
\cdots & \xrightarrow{p} & F(\Delta_F \Delta X) & \xrightarrow{p} & F(\Delta X) & \xrightarrow{p} & X
\end{array}$$

This diagram induces a map from the inverse limit of the top sequence, which is just  $|\Delta X|$  since all the maps are homeomorphisms, to the inverse limit of the bottom sequence, which we call  $\Delta^\infty X$ . Corollary 4.13 says that up to homotopy, any map from the top of this diagram to a locally finite space factors through the bottom of the diagram if you go far out enough to the left. Translated into a statement about the inverse limits, this implies that any map from  $|\Delta X|$  to a locally finite space factors through  $\Delta^\infty X$  up to homotopy.

In fact, this map  $i = \pi_\infty : |\Delta X| \rightarrow \Delta^\infty X$  has some even stronger properties. It is not too difficult to show that  $i$  is an embedding whose image is exactly the set of closed points of  $\Delta^\infty X$ , which is a dense subspace. The closure of every point contains exactly one closed point, and mapping each point to that unique closed point defines a continuous left inverse  $r$  for  $i$ . In fact,  $ir$  is homotopic to the identity on  $\Delta^\infty X$  via a step homotopy, so  $|\Delta X|$  is a deformation retract of  $\Delta^\infty X$  under  $i$ . Thus when we take the inverse limit of the weak homotopy equivalences  $\pi_n$ , we actually obtain a strong homotopy equivalence. As the proof of all this is fairly involved and rather tangential to our main interest

here, we leave the details as an exercise for the reader with an good taste for pointset topology. We recommend first looking at the case when  $X$  is a totally ordered set with two elements, which is relatively simple but illustrative of the general case.

There is also a version of simplicial approximation for homotopies, rather than just single maps. First, we recall the following definition.

**Definition 4.14.** *Let  $X$  and  $Y$  be simplicial complexes and  $f, g : X \rightarrow Y$  be simplicial maps. Then  $f$  and  $g$  are contiguous if for every cell  $A$  of  $X$ ,  $f(A) \cup g(A)$  is a cell of  $Y$ . If there is a sequence  $f = h_1, h_2, \dots, h_n = g$  of simplicial maps such that  $h_i$  is contiguous to  $h_{i+1}$  for each  $i$ , we say  $f$  and  $g$  are contiguous-equivalent.*

It is easy to see that contiguous maps are always homotopic via a linear homotopy. Conversely, there is the following analog of the simplicial approximation theorem above.

**Theorem 4.15.** *Let  $X$  and  $Y$  be finite simplicial complexes and  $f, g : X \rightarrow Y$  be simplicial maps that are homotopic. Then for some  $n$ ,  $\Delta_F(f)\Delta(p^{n-1}) : \Delta_F^n X \rightarrow \Delta_F^n Y$  and  $\Delta_F(g)\Delta(p^{n-1}) : \Delta_F^n X \rightarrow \Delta_F^n Y$  are contiguous-equivalent, where  $\Delta(p^{n-1}) : \Delta_F^n X \rightarrow \Delta_F^n X$  is the simplicial map given by the canonical map  $p^{n-1} : F(\Delta_F^{n-1} X) \rightarrow F(X)$  of finite spaces.*

*Proof.* This follows from Lemma 3.5.4 and Theorem 3.5.6 of [10], since  $\Delta(p^{n-1})$  is a simplicial approximation to the identity so  $\Delta_F(f)\Delta(p^{n-1})$  and  $\Delta_F(g)\Delta(p^{n-1})$  are simplicial approximations to the geometric realizations of  $f$  and  $g$ .  $\square$

In order to translate this into a statement about finite spaces, we first have to relate contiguity to homotopy on finite spaces.

**Lemma 4.16.** *Let  $X$  and  $Y$  be finite spaces and  $f, g : X \rightarrow Y$ . If  $f$  and  $g$  are step-homotopic, then the induced simplicial maps  $\Delta f$  and  $\Delta g : \Delta X \rightarrow \Delta Y$  are contiguous-equivalent. Conversely, if  $\Delta f$  and  $\Delta g$  are contiguous,  $f$  and  $g$  are homotopic.*

*Proof.* Suppose  $f \leq g$ ; the down-homotopy case is similar. Let  $\leq$  be a total order on  $X$  extending the partial order, and write  $X = \{x_i\}$  with  $x_n \prec x_{n-1} \dots \prec x_0$ . Let  $f_k : X \rightarrow Y$  be given by  $f_k(x_j) = f(x_j)$  for  $j \geq k$  and  $f_k(x_j) = g(x_j)$  for  $j < k$ . Then since  $f \leq g$  and  $x \leq y$  implies  $x \leq y$ , it is clear that each  $f_k$  is order-preserving and  $f_k \leq f_{k+1}$  for each  $k$ . Since  $f_0 = f$  and  $f_{n+1} = g$ , it suffices to show that  $\Delta f_k$  is contiguous to  $\Delta f_{k+1}$  for each  $k$ . Now if  $C$  is a cell of  $\Delta X$  (i.e., a chain in  $X$ ) and  $x_k \notin C$ ,  $\Delta f_k(C) = \Delta f_{k+1}(C)$  so  $\Delta f_k(C) \cup \Delta f_{k+1}(C) = \Delta f_k(C)$  is a cell of  $\Delta Y$  since  $f_k$  is order-preserving. If  $x_k \in C$ , then for any  $x < x_k$  in  $C$ ,  $f_k(x) = f_{k+1}(x) \leq f_{k+1}(x_k)$ , and similarly if  $x > x_k$  then  $f_k(x) = f_{k+1}(x) \geq f_{k+1}(x_k)$ . Thus since  $\Delta f_k(C)$  is a chain, so is  $\Delta f_k(C) \cup \{f_{k+1}(x_k)\} = \Delta f_k(C) \cup \Delta f_{k+1}(C)$ . Hence  $f_k$  and  $f_{k+1}$  are contiguous.

Conversely, suppose  $\Delta f$  and  $\Delta g$  are contiguous. Then for any  $x \in X$ ,  $\Delta f(\{x\}) \cup \Delta g(\{x\}) = \{f(x), g(x)\}$  is a chain in  $Y$ , so  $f(x)$  and  $g(x)$  are comparable. Let  $h(x) = \max(f(x), g(x))$ . Then  $h$  is clearly order-preserving,  $f \leq h$ , and  $h \geq g$ . Thus  $f$  and  $g$  are homotopic via a finite-homotopy through  $h$ .  $\square$

Note that this does not imply that  $f$  and  $g$  are homotopic iff  $\Delta f$  and  $\Delta g$  are contiguous-equivalent, because  $\Delta f$  and  $\Delta g$  might be contiguous-equivalent through simplicial maps that are not order-preserving. However, if we apply  $F$  instead of “removing”  $\Delta$ , simplicial maps do become order-preserving, so if  $\Delta f$  and  $\Delta g$  are contiguous-equivalent, it follows that  $F(\Delta f)$  and  $F(\Delta g)$  are homotopic.

**Theorem 4.17.** *Let  $f, g : X \rightarrow Y$  where  $X$  is finite and  $Y$  is locally finite, and suppose  $|\Delta f|$  and  $|\Delta g|$  are homotopic. Then for some  $n$ ,  $fp^n : F(\Delta_F^{n-1}\Delta X) \rightarrow Y$  and  $gp^n : F(\Delta_F^{n-1}\Delta X) \rightarrow Y$  are homotopic, where  $p^n : \Delta^n X \rightarrow X$  is the canonical map.*

*Proof.* As in the proof of Theorem 4.12, we may assume  $Y$  is actually finite. Since the simplicial maps  $\Delta f$  and  $\Delta g$  on the order complexes are homotopic, by Theorem 4.15  $\Delta_F \Delta(f) \Delta_F \Delta(p)^n$  and  $\Delta^2(g) \Delta^2(p)^n$  are contiguous-equivalent for some  $n$ . By the discussion following Lemma 4.16, this means that  $F(\Delta_F \Delta(f)) F(\Delta_F \Delta(p)^n) = F \Delta_F \Delta(fp^n)$  and  $F(\Delta_F \Delta(gp^n))$  are homotopic. Now for  $p_Y^2 : F(\Delta_F \Delta Y) \rightarrow Y$  the canonical map, naturality of  $p$  implies that  $p_Y^2 F(\Delta_F \Delta(fp^n)) = fp^{n+2}$ , and similarly for  $g$ . Thus  $fp^{n+2}$  and  $gp^{n+2}$  are homotopic since  $F(\Delta_F \Delta(fp^n))$  and  $F(\Delta_F \Delta(gp^n))$  are.  $\square$

While it is easy to get lost in the details of these arguments, they are really just formal manipulations to translate the simplicial approximation theorem from the language of simplicial complexes to the language of finite spaces.

#### 4.4 A finitistic “singular” homology for finite spaces

While the lifting theorem allows us to compute the singular homology of finite spaces easily, one might wonder whether there is a natural definition of homology that remains completely within the category of finite spaces. A moment’s reflection shows that it is easy to define simplicial homology in this way: an  $n$ -simplex in the order complex  $\Delta X$  is exactly a chain  $C \subseteq X$  of length  $n + 1$ , which is the same as an injective map  $D^n \rightarrow X$ , where  $D^n$  is a totally ordered set with  $n + 1$  elements. Indeed,  $\Delta D^n$  is isomorphic as a simplicial complex to the  $n$ -simplex, so  $D^n$  is the natural model of an  $n$ -simplex in the category of finite spaces. Simplicial homology on finite spaces is then the homology of the complex  $C_n^\Delta(X) = \{\text{formal linear combinations of injective maps } D^n \rightarrow X\}$ .

The natural definition of “singular” homology in the category of finite spaces would thus be to consider chains built from *all* maps  $D^n \rightarrow X$ , rather than just injective maps. Thus for a finite space  $X$  and  $n \geq 0$ , we let  $C_n^f(X)$  be the free abelian group on the set of maps  $D^n \rightarrow X$ . For  $c : D^n \rightarrow X$ , we define  $\partial c \in C_{n-1}^f(X)$  by  $\partial c = \sum_k (-1)^k c|_{D^n - \{x_k\}}$ . Extending  $\partial$  linearly to  $C_n^f(X)$ , we then obtain a chain complex which we call the *finite-singular complex* of  $X$ . We denote the homology of this complex by  $H_n^f(X)$  and call it the *finite-singular homology* of  $X$ . If we augment the finite-singular complex to include  $C_{-1}^f(X) = \mathbb{Z}$  (generated by the set of maps from  $D^{-1}$ , the empty space, to  $X$ ), we obtain  $\tilde{H}_n^f(X)$ , the reduced finite-singular homology of  $X$ . For  $A \subseteq X$ , the

relative finite-singular homology  $H_n^f(X, A)$  is defined as for singular homology. Everything defined here is functorial in the obvious way.

We now show that finite-singular homology is naturally isomorphic to singular homology. We note that composing with the quotient map  $|\Delta D^n| \rightarrow D^n$  turns any finite-singular simplex into an actual singular simplex, so there is a natural complex map  $i : C_n^f(X) \rightarrow C_n(X)$ , for  $C_n(X)$  the singular chain complex of  $X$ . We will identify  $D^n$  with  $\{0, 1, \dots, n\}$ .

**Lemma 4.18.** *Let  $X$  be a finite space with a greatest element  $x$ . Then  $\tilde{H}_n^f(X) = 0$  for all  $n$ .*

*Proof.* For  $c : D^n \rightarrow X$  and  $i \in D^{n+1}$ , define  $c_i : D^{n+1} \rightarrow X$  by  $c_i(j) = c(j)$  for  $j < i$  and  $c_i(j) = x$  for  $j \geq i$ . Then each  $c_i$  is continuous since  $x$  is the greatest element of  $X$ . We write  $Hc = \sum_{i=1}^{n+1} (-1)^i c_i$ , and extend to  $H : C_n^f(X) \rightarrow C_{n+1}^f(X)$  linearly. Now let  $\alpha^n : D^n \rightarrow X$  be the constant map sending every point to  $x$ . If we calculate  $\partial Hc$ , we get a signed sum of maps  $D^n \rightarrow X$  obtained by appending  $x$  to  $c$  and replacing a tail end of  $c$  by  $x$  and then dropping one point. Similarly,  $H\partial c$  is a signed sum of maps obtained by dropping a point from  $c$  and then appending  $x$  and replacing a tail end (but not the entire chain) with  $x$ . It is not difficult to check that these two sums are equal, except that they have opposite sign and  $\partial Hc$  has extra terms  $-\alpha^n$  and  $c$  that are missing from  $H\partial c$ . That is,  $\partial Hc + H\partial c = c - \alpha^n$ . By linearity, we obtain that  $\partial Hc + H\partial c = c - k\alpha^n$  for some  $k \in \mathbb{Z}$  for all  $c \in C_n^f(X)$ .

Now suppose  $[c] \in \tilde{H}_n^f(X)$ . Then  $\partial c = 0$ , so  $\partial Hc = c - k\alpha^n$  and hence  $[c] = k[\alpha^n]$ . If  $n$  is even,  $\partial\alpha^n = \alpha^{n-1}$ , so the only way  $k\alpha^n$  can be a cycle is if  $k = 0$ . If  $n$  is odd,  $\alpha^n = \partial\alpha^{n+1}$ , so  $\alpha^n$  is a boundary. Either way,  $[c] = k[\alpha^n]$  implies that  $[c] = 0$ . Hence  $\tilde{H}_n^f(X) = 0$ , as desired.  $\square$

**Lemma 4.19.** *Finite-singular homology has Mayer-Vietoris sequences for any decomposition of a finite space  $X = U \cup V$  as the union of two open subspaces. Also,  $i_* : H_n^f(X) \rightarrow H_n(X)$  commutes with the homomorphisms of Mayer-Vietoris sequences.*

*Proof.* The only part of the usual proof that Mayer-Vietoris sequences exist that is not pure algebra is the fact that  $C_n^f(X)$  is generated by  $C_n^f(U) \cup C_n^f(V)$ . Now for any map  $c : D^n \rightarrow X$ ,  $c(n)$  is either in  $U$  or  $V$ . Since  $U$  and  $V$  are downsets and  $c$  is order-preserving, this implies that the entire image of  $c$  is in either  $U$  or  $V$ . It follows that  $C_n^f(U) \cup C_n^f(V)$  generates all of  $C_n^f(X)$ . The statement about  $i_*$  follows from the fact that  $i$  induces a morphism from the short exact sequence of complexes defining the finite-singular Mayer-Vietoris sequence to the short exact sequence of complexes defining the singular Mayer-Vietoris sequence.  $\square$

**Theorem 4.20.** *For pairs of finite spaces, the relative homology functors  $H_n^f$  and  $H_n$  are naturally isomorphic via the induced map  $i_*$  from the complex map  $i : C_n^f \rightarrow C_n$ .*

*Proof.* By the five lemma, it suffices to show that  $i_*$  is an isomorphism for absolute (rather than relative) homology. Naturality of  $i_*$  is obvious from the

way it is defined. Since it is easy to see that both  $H_n^f$  and  $H_n$  differ from their reduced versions only in that the 0th homology has an extra  $\mathbb{Z}$  direct summand and this  $\mathbb{Z}$  summand is preserved by  $i_*$ , it suffices to show that for any finite space  $X$ ,  $i_* : \tilde{H}_n^f(X) \rightarrow \tilde{H}_n(X)$  is an isomorphism for all  $n$ .

We use induction on  $|X|$ ; the case  $|X| = 0$  is trivial. Now suppose  $|X| > 0$  and let  $x \in X$  be a maximal point. Let  $U = x^\perp$  and  $V = X - \{x\}$ . Then  $|V| < |X|$ , so by induction,  $i_*$  induces isomorphisms for  $V$  and  $V \cap U$ . By Lemma 4.18, the reduced finite-singular homology of  $U$  is trivial, and so is the reduced singular homology since  $U$  is contractible. Thus  $i_*$  induces isomorphisms for  $U$ ,  $V$ , and  $U \cap V$ . By Lemma 4.19,  $i_*$  commutes with the Mayer-Vietoris sequence for  $U$  and  $V$ , so by the five lemma  $i_*$  also induces isomorphisms for  $U \cup V = X$ .  $\square$

We note that finite-singular homology can also be defined for locally finite spaces. By the usual compactness argument, the finite-singular homology of a locally finite space is the direct limit of the finite-singular homologies of its finite subspaces, and the same holds for singular homology. It follows that Theorem 4.20 is also valid for locally finite spaces.

If we consider finite-singular homology as being defined not on a finite space but on its order complex, the definition extends in an obvious way to arbitrary simplicial complexes. We leave it as an exercise for the reader to show that Theorem 4.20 then holds for all simplicial complexes, not just order complexes.

## 5 Generalizations and other approaches

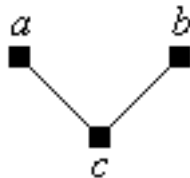
In this section we reexamine some of the earlier definitions and theorems and compare our approach with historical approaches. In particular, we note that many of the constructions throughout this paper do not quite work as well as one might hope for infinite locally finite spaces (or just infinite order spaces), and we examine the extent to which things work better if you use alternate approaches.

### 5.1 Generalizations and counterexamples to the lifting theorem

First, we look at how one might weaken the hypotheses of the lifting theorem. Recall the notation of Theorem 3.5: we have a function  $f : X \rightarrow Y$  from a perfectly normal space to a finite space and want to lift it to a map  $\tilde{f} : X \rightarrow |\Delta Y|$  (for simplicity, we work with the case  $A = \emptyset$ ). We choose a maximal element  $y \in Y$  and let  $V = f^{-1}(Y - \{y\})$ . The proof of the theorem then (essentially) uses perfect normality to turn a lift  $f_0 : V \rightarrow |\Delta(Y - \{y\})|$  of  $f|_V$ , which exists by induction, into a lift of  $f$ . What in this construction can be weakened?

First, the hypothesis that  $X$  be perfectly normal is absolutely necessary: if  $C \subseteq X$  is closed, then  $f : C \mapsto 1, X - C \mapsto 0$  is a continuous map  $X \rightarrow D^1$ , and any lift  $\tilde{f} : X \rightarrow \Delta D^1 = I$  would satisfy  $\tilde{f}^{-1}(\{1\}) = C$ . A slightly more

Figure 9:  $Y = \{a, b, c\}$



complicated construction can be used to show that  $X$  must also be normal. Thus if the lifting theorem always holds for  $X$ ,  $X$  must be perfectly normal.

The hypothesis that  $Y$  be finite, however, seems less crucial. In fact, perhaps  $Y$  need not even be locally finite: the definition of the order complex and the quotient map on it makes sense for any poset. At the very least, if  $Y$  is a well-founded ordered set, it seems plausible that the inductive construction of Theorem 3.5 might generalize.

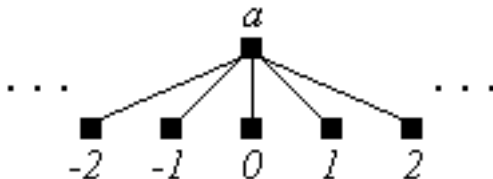
However, there is one key difficulty in iterating the induction of Theorem 3.5 infinitely many times: in order to extend our lift  $f_0$  of  $f|_V$ , we first had to modify it slightly (to obtain what we called  $f_1$ ). The reason for this is illustrated by the following example. Let  $Y = \{a, b, c\}$  as shown in Figure 9 and define  $f : I \rightarrow Y$  by  $f(0) = a$ ,  $f|_{(0,1)} = b$ , and  $f(1) = c$ . Suppose we are constructing a lift of  $f$  by starting with a lift  $f_0$  of  $f|_{\{a,b\}}$ . We identify  $|\Delta(\{a, b\})|$  with  $I$ , with 1 mapping to  $a$  and  $[0, 1)$  mapping to  $b$ . Now we obtain  $|\Delta Y|$  from  $I$  by adding a cone on  $\{0\}$ , and our lift of  $f$  must map 1 to the vertex of this cone. However, there is no reason that 0 even has to be in the range of  $f_0$  (eg,  $f_0(t) = 1 - t/2$ ). In order to make  $\tilde{f}$  continuous while having  $\tilde{f}(1)$  be the vertex of the cone, we want to have  $\tilde{f}(t)$  grow out in the cone dimension as  $t$  approaches 1. But the cone dimension doesn't even exist at points where  $f_0 > 0$ , so this is impossible unless we first modify  $f_0(t)$  to be 0 near  $t = 1$ .

Since at each stage of the induction, we modify the function we had before rather than just extending it, if we iterate the induction infinitely many times, the end result may not be well-defined. Indeed, given a point  $x \in X$ , every time we add a point  $y$  to our range such that  $y > f(x)$ , the value of our partial lift at  $x$  might change. However, this means that if there are only finitely many points above  $f(x)$ , the value of the partial lift at  $x$  is eventually stable. This is true for all  $x$  iff  $Y$  is bilocally finite. Since connected bilocally finite spaces are countable, we can then just use ordinary induction to construct our lift. To see that the lift we get in the limit is continuous, note that the domain of each partial lift is open and the partial lifts are eventually constant in a neighborhood of each point. Thus we have:

**Theorem 5.1** (Lifting Theorem for Bilocally Finite Spaces). *The statement of Theorem 3.5 holds when  $Y$  is assumed only to be bilocally finite.*

One might ask whether it might somehow be possible to salvage the induction argument to make it work for arbitrary locally finite spaces, or at least some

Figure 10:  $Y = \{a\} \cup \mathbb{Z}$



larger class of order spaces. One idea is instead of building  $Y$  from the bottom up in the induction, we can build it from the top down. Indeed, the inductive description of  $\delta Y = |\Delta Y|$  at the beginning of Section 3.1 is also valid if we build  $Y$  from the top down. The spaces that we can obtain “from the top down” by repeatedly adding minimal points will not be locally finite spaces but colocally finite spaces (i.e.,  $Y^{\text{op}}$  is locally finite). With some modification (actually, mostly simplification), the proof of Theorem 3.5 can be adapted to do induction from the top down, and it turns out that in this case, it is *not* necessary to modify our previous partial lift. Thus if we do an infinite induction, the total lift to a colocally finite space will always be well-defined.

However, a new problem arises with colocally finite spaces. Because we’ve essentially reversed the order of everything, the domains of the partial lifts will be closed, not open. Thus it is not clear that the total function we obtain will be continuous. In fact, in general this does fail. For example, consider the infinite space  $Y = \{a\} \cup \mathbb{Z}$  with the ordering shown in Figure 10. We identify  $|\Delta Y|$  with  $I \times \mathbb{Z} / \{0\} \times \mathbb{Z}$  in the natural way; note that then  $\pi^{-1}(\{n\}) = (0, 1] \times \{n\}$  for each  $n$  and  $\pi^{-1}(\{a\})$  is just the wedge point. Define  $f : I \rightarrow Y$  by  $f|_{(1/n, 1/(n+1))} = n$  and  $f(t) = a$  for all other  $t$ . Then if  $\tilde{f}$  is a lift of  $f$ , since  $f$  is surjective, the image of  $\tilde{f}$  must intersect every  $(0, 1] \times \{n\}$ ; let  $\epsilon_n$  be such that  $(\epsilon_n, n)$  is in the image of  $\tilde{f}$ . Then  $U = \bigcup [0, \epsilon_n) \times \{n\}$  is an open set in  $|\Delta Y|$  that by hypothesis does not contain all of  $(1/n, 1/(n+1))$  for any  $n$ . But  $U$  does contain the wedge point, so in particular  $0 \in \tilde{f}^{-1}(U)$ . However, any neighborhood of  $0$  must contain  $(1/n, 1/(n+1))$  for all  $n$  sufficiently large. Hence  $\tilde{f}^{-1}(U)$  cannot be open, so  $\tilde{f}$  cannot be continuous.

We leave it to the reader to construct a similar counterexample when  $Y$  is the set of negative integers with the usual ordering. This shows that the lifting theorem fails for colocally finite spaces whether they are infinite because they are “fat” or because they are “tall”. Furthermore, in these counterexamples  $X$  is just  $I$ , an extremely nice space. It is worth noting that these counterexamples work precisely because the topology on infinite simplicial complexes is extremely fine, and the lifting theorem requires the construction of maps into these complexes. Perhaps then the solution is to replace the “weak” simplicial complex topology with, say, the natural product topology obtained by considering the complex as a subspace of  $\mathbb{R}^V$  for  $V$  the vertex set. While it does seem that the lifting theorem would work with respect to this topology, the usefulness of such

a theorem is doubtful, since such infinite-dimensional spaces seem no easier to compute with than the colocally finite spaces themselves.

We thus set aside colocally finite spaces and turn back to locally finite spaces. By Corollary 3.10, any counterexample must be considerably more exotic than the ones above for colocally finite spaces. Essentially the only way to construct a counterexample I know of is to take a space  $X$  that is perfectly normal but which contains a closed discrete subspace  $D = \{x_\alpha\}$  such that there is no collection of disjoint open sets  $U_\alpha$  such that  $x_\alpha \in U_\alpha$  for each  $\alpha$ .<sup>3</sup> Note that such a  $D$  is necessarily uncountable. Now letting  $Y = \{a\} \cup \{x_\alpha\}$  ordered by  $x_\alpha > a$  for all  $\alpha$ , the map  $f : X \rightarrow Y$  sending each  $x_\alpha$  to itself and  $X - D$  to  $a$  is continuous. However, identifying  $\downarrow \Delta Y$  with  $I \times D / \{0\} \times D$  in the obvious way, for any lift  $\tilde{f}$  the open sets  $U_\alpha = \tilde{f}^{-1}((0, 1] \times \{x_\alpha\})$  would be disjoint with  $x_\alpha \in U_\alpha$  for each  $\alpha$ .

Thus the lifting theorem does fail for arbitrary perfectly normal  $X$  and arbitrary locally finite  $Y$ . However, this sort of counterexample can only work if  $Y$  is uncountable, and it seems plausible that the lifting theorem might hold for countable locally finite spaces. In any case, for virtually all practical purposes Corollary 3.10 is sufficient for locally finite spaces, and by the discussion above it is unlikely that a version of the lifting theorem would hold for nonlocally finite spaces (but see Corollary 5.6 below).

## 5.2 McCord and the homotopy-theoretic approach

We now turn to a totally different approach to finite spaces: the homotopy-theoretic approach of McCord in [7], the paper in which Corollary 3.7 first appeared. Interestingly, McCord's own original method was different and this approach was suggested by the referee; unfortunately, I have been unable to find any information about the original method. This approach is based on applying very general homotopy-theoretic tools for showing that maps are weak homotopy equivalences. Among the better-known results of this sort are the nerve lemma and the Quillen fiber lemma. McCord himself proved the following modification of a theorem of Dold and Thom.

**Theorem 5.2** ([7], Theorem 6). *Let  $X$  and  $Y$  be spaces and  $f : X \rightarrow Y$ . Suppose there is an open cover  $\mathcal{U}$  of  $Y$  such that  $f|_{f^{-1}(U)} : f^{-1}(U) \rightarrow U$  is a weak homotopy equivalence for each  $U \in \mathcal{U}$  and such that  $\mathcal{U}$  is a basis for a topology (i.e., for any  $U, V \in \mathcal{U}$ ,  $U \cap V = \bigcup \{W \in \mathcal{U} : W \subseteq U \cap V\}$ ). Then  $f$  is a weak homotopy equivalence.*

If  $X$  is finite and  $x \in X$ , both  $x^\downarrow$  and  $\pi^{-1}(x^\downarrow)$  are contractible. Since sets of the form  $x^\downarrow$  clearly form a basis, the theorem then implies Corollary 3.7. However, this argument works just as well for arbitrary posets as it does for finite spaces. Thus we obtain the following considerable generalization of Corollary 3.7.

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<sup>3</sup>See Example H in [5] for a construction of such a space; this space is a small modification of Example 142 in [11].



**Corollary 5.3** ([7], Theorem 2). *Let  $X$  be a poset. Then the quotient map  $\pi : |\Delta X| \rightarrow X$  is a weak homotopy equivalence.*

As an immediate corollary, we generalize Corollary 3.9 to arbitrary posets (in particular, locally finite non-bilocally finite spaces).

**Corollary 5.4.** *Let  $X$  be a poset. Then  $X$  and  $X^{\text{op}}$  are weak homotopy equivalent.*

Interestingly, while a priori Theorem 3.5 (or Corollary 3.10) is a much stronger statement than Corollary 3.7, and indeed Corollary 5.3 together with the counterexamples in the previous section show that it is in some sense strictly stronger, McCord notes that Corollary 5.3 implies at least a form of the lifting theorem mod homotopy. He states the following result without proof; my proof is adapted from the proof of Lemma 4.6 of [6].

**Theorem 5.5** ([8], Lemma 1). *Let  $p : X \rightarrow Y$  be a weak homotopy equivalence,  $Z$  be a CW-complex, and  $f : Z \rightarrow Y$ . Then there is a map  $\tilde{f} : Z \rightarrow X$  such that  $p\tilde{f}$  is homotopic to  $f$ .*

*Proof.* It is easy to see that such a  $\tilde{f}$  is equivalent to a homotopy from  $f$  to a map  $\tilde{f} : Z \rightarrow X \subseteq M_p$  in the mapping cylinder  $M_p$  of  $p$ . Now since  $p$  is a weak homotopy equivalence, the relative homotopy groups  $\pi_n(M_p, X)$  are all trivial. We construct our homotopy cell-by-cell: if  $f$  has already been homotoped into  $DY$  on the boundary  $\partial B^n$  of a cell  $B^n$  of  $Z$ , then  $f : (B^n, \partial B^n) \rightarrow (M_p, X)$  defines an element of  $\pi_n(M_p, X)$ . Since  $\pi_n(M_p, X) = 0$ , this means that  $f|_{\partial B^n}$  can be homotoped rel  $\partial B^n$  to a map whose image is contained in  $B^n$ . By the homotopy extension property, this homotopy on  $B^n$  (and whatever else we have already done inductively) extends to all of  $X$ .  $\square$

**Corollary 5.6.** *Let  $Y$  be a poset,  $X$  be a CW-complex, and  $f : X \rightarrow Y$ . Then there is a map  $\tilde{f} : X \rightarrow |\Delta Y|$  such that  $\pi\tilde{f}$  is homotopic to  $f$ .*

It is not hard to see that this result could be modified to require  $\tilde{f}$  to extend a partial lift of  $f$  on a subcomplex of  $X$ . As in Theorem 3.5, this can be used to show that  $\tilde{f}$  is unique up to homotopy. McCord himself uses this result in [8] to prove a stronger version of Corollary 5.3. Given a space  $X$  with an open covering  $\mathcal{U}$  sufficiently similar to the open covering of a poset by the downsets of each of its points, he defines  $p : X \rightarrow \mathcal{U}$  by mapping  $x \in X$  to the smallest  $U \in \mathcal{U}$  such that  $x \in U$ ; this map is continuous when  $\mathcal{U}$  is considered as a poset ordered by inclusion. We note the similarity between this and the definition of  $\pi : |\Delta \mathcal{U}| \rightarrow \mathcal{U}$ , which is indeed a special case of this. Theorem 5.2 applied to the open cover of  $\mathcal{U}$  by the downsets of points then implies that  $p$  is a weak homotopy equivalence. Theorem 5.5 then says that  $\pi : |\Delta \mathcal{U}| \rightarrow \mathcal{U}$  lifts mod homotopy to a map  $\tilde{\pi} : |\Delta \mathcal{U}| \rightarrow X$ , which is easily seen to be a weak homotopy equivalence. Thus Corollary 5.3 holds not only for posets, but for spaces that look enough like a poset.

### 5.3 Other directions and historical notes

Posets as topological spaces were first analyzed by Alexandroff in [1]. They can more abstractly be characterized as the spaces in which an arbitrary intersection of open sets is open, and Theorems 1.3 and 1.5 generalize to say that the association of an arbitrary poset to its order space is a full faithful embedding of the category of posets in the category of spaces. Alexandroff called order spaces “discrete spaces,” but since this term now has a different meaning, they are often called “Alexandroff spaces” (McCord calls them “A-spaces”).

Alexandroff also defined the order complex of a poset, and observed at least in a special case that the order complex has the same homology as the poset (as we noted, this can be easily proven using Mayer-Vietoris sequences). McCord considerably extended this in [7] with Theorem 5.3, and also proved Proposition 1.1 to include non- $T_0$  finite spaces in the analysis. At the same time, Stong analyzed finite spaces from a more combinatorial point of view in [12]. Stong discovered most of the results in this paper that do not deal with order complexes, though except for Section 4.2, I discovered them independently. Stong called rigidifications “cores,” and constructed them as subspaces rather than as quotients. Among the results in [12] not in this paper are an analysis of finite H-spaces and a version of Proposition 4.5 for maps from a finite simplicial complex to a finite space.

In a series of recent papers, Barmak and Minian have explored various topics in the algebraic topology of finite spaces. In [2], they studied spaces of minimal cardinality of a given weak homotopy type and proved in particular that  $S_f^n$  is the minimal “finite model” of  $S^n$  for all  $n$ , a fact which intuitively seems very likely. In [3], they characterize simple homotopy of simplicial complexes in the language of finite spaces. That paper also provides what is as far as I know the first example in print of a finite space that is weak homotopy equivalent to a point but not contractible, although such spaces are quite easy to construct using Corollary 4.10. Finally, [4] further explores simple homotopy on finite spaces and proves that, like simplicial complexes, regular CW-complexes are weak homotopy equivalent to their posets of cells (in fact, slightly less than regularity is needed).

Order complexes are well-known among topological combinatorialists, though their relation to finite spaces less so. Many naturally occurring simplicial complexes are order complexes, and the additional order structure often makes order complexes easier to work with than arbitrary simplicial complexes. For example, a *shelling* on a simplicial complex is an ordering of its cells which satisfies certain nice properties, which in particular imply that the complex is homotopy equivalent to a wedge of spheres. For order complexes, however, instead of having to order the entire set of cells, it suffices to give a certain kind of labelling of the edges in the Hasse diagram of the poset. For a survey of this and many other applications of order complexes in combinatorics, see [13].

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